The first higher Stasheff–Tamari orders are quotients of the higher Bruhat orders

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Formal Power Series and Algebraic Combinatorics

Overview

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In [Wil21], we prove their conjecture.

The higher Bruhat orders

Given $A \in \binom{[n]}{\delta+2}$, *packet* of A: $P(A) := \{B \mid B \in \binom{[n]}{\delta+1}, B \subset A\}$.

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$\mathcal{B}(n, \delta+1)$: partial order on equiv. classes of admissible orders of $\binom{[n]}{\delta+1}$, with covering relations $[\alpha] \lessdot [\alpha']$ iff $\mathrm{inv}(\alpha') = \mathrm{inv}(\alpha) \cup \{A\}$ where $A \in \binom{[n]}{\delta+2} \setminus \mathrm{inv}(\alpha)$ [MS89].

Higher Bruhat orders: examples

Example

Equivalence class representatives of the elements of $\mathcal{B}(4,2)$ are:

$$\begin{aligned}
\hat{0} &= \{12 < 13 < 14 < 23 < 24 < 34\}, \\
\alpha_1 &= \{23 < 13 < 12 < 14 < 24 < 34\}, \\
\alpha_2 &= \{23 < 24 < 13 < 14 < 34 < 12\}, \\
\alpha_3 &= \{23 < 24 < 34 < 14 < 13 < 12\}, \\
\beta_1 &= \{12 < 13 < 14 < 34 < 24 < 34\}, \\
\beta_2 &= \{34 < 12 < 14 < 13 < 24 < 23\}, \\
\beta_3 &= \{34 < 24 < 14 < 12 < 13 < 23\}, \\
\hat{1} &= \{34 < 24 < 23 < 14 < 13 < 12\}.
\end{aligned}$$

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The inversion sets are:

$$
\begin{aligned}
\mathrm{inv}(\hat{0}) &= \varnothing, \\
\mathrm{inv}(\alpha_1) &= \{123\}, \\
\mathrm{inv}(\alpha_2) &= \{123, 124\}, \\
\mathrm{inv}(\alpha_3) &= \{123, 124, 134\}, \\
\mathrm{inv}(\beta_1) &= \{234\}, \\
\mathrm{inv}(\beta_2) &= \{134, 234\}, \\
\mathrm{inv}(\beta_3) &= \{124, 134, 234\}, \\
\mathrm{inv}(\hat{1}) &= \{123, 124, 134, 234\}.
\end{aligned}
$$

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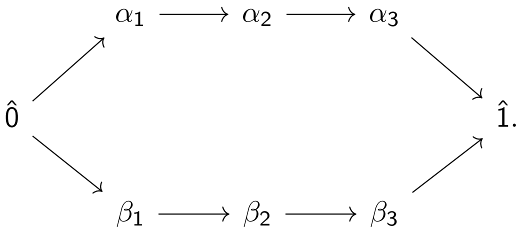
Hence the poset is:



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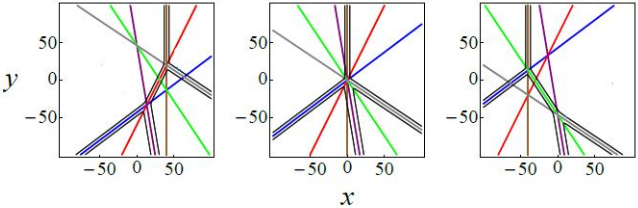

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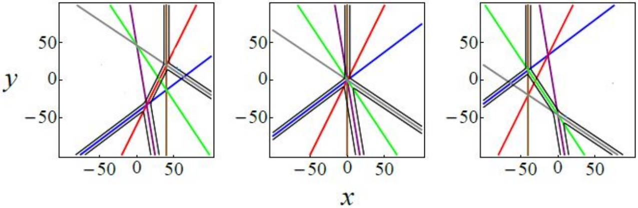

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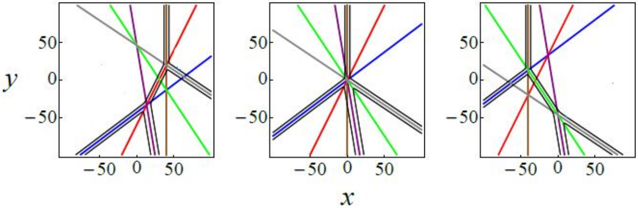

There is a hyperplane arrangement in the background, given by equality of pairs of phases. Different hyperplane arrangements correspond to elements of the higher Bruhat orders.

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$V(\alpha)$ denotes the elements of $\binom{[n]}{\delta+1}$ which are visible in α.

The *higher Tamari order* $\mathcal{T}(n, \delta+1)$ has elements $\{V(\alpha) \mid [\alpha] \in \mathcal{B}(n, \delta+1)\}$ with $V(\alpha) \leqslant V(\alpha')$ iff $[\alpha] \leqslant [\alpha']$.

Higher Tamari orders: example

Example

Getting rid of invisible elements:

$$\begin{aligned}
V(\hat{0}) &= \{12 < 23 < 34\}, \\
V(\alpha_1) &= \{13 < 34\}, \\
V(\alpha_2) &= \{13 < 34\}, \\
V(\alpha_3) &= \{14\}, \\
V(\beta_1) &= \{12 < 24\}, \\
V(\beta_2) &= \{12 < 24\}, \\
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V(\hat{1}) &= \{14\}.
\end{aligned}$$

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Hence the higher Tamari poset $\mathcal{T}(4,2)$ is:

$$\begin{array}{ccccc} & & \{13 < 34\} & & \\ & \nearrow & & \searrow & \\ \{12 < 23 < 34\} & & & & \{14\}. \\ & \searrow & & \nearrow & \\ & & \{12 < 24\} & & \end{array}$$

The (first) higher Stasheff–Tamari orders

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A triangulation $\mathcal{T}'$ of $C(n, \delta)$ is an *increasing bistellar flip* of a triangulation $\mathcal{T}$ if it is the result of replacing the lower triangulation of a copy of $C(\delta + 2, \delta)$ within $\mathcal{T}$ by the upper triangulation.

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The (first) higher Stasheff–Tamari order $\mathcal{S}(n, \delta)$ is the poset on triangulations of $C(n, \delta)$ with covering relations given by increasing bistellar flips [KV91; ER96].

Cubillages of cyclic zonotopes

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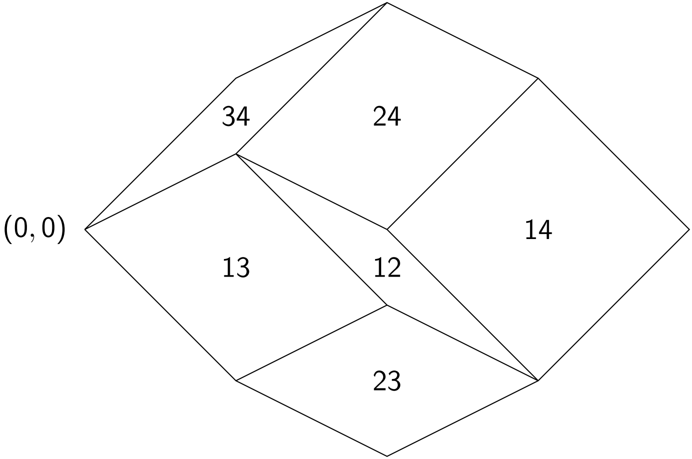
The higher Bruhat order $\mathcal{B}(n, \delta + 1)$ is the poset on cubillages of $Z(n, \delta + 1)$ with covering relations given by increasing flips [KV91; Tho03].

Cubillages: example



Visibility in terms of cubillages

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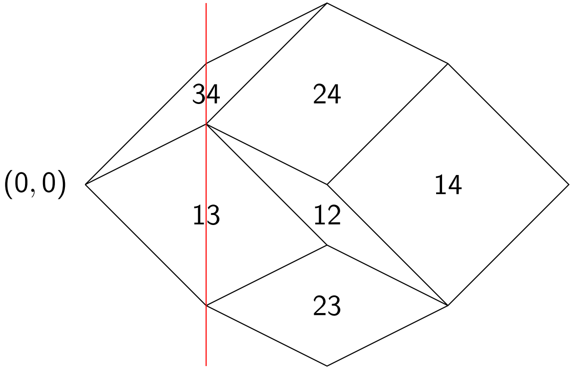
Hence, taking the visible elements of a cubillage gives us an order-preserving map

$$V : \mathcal{B}(n, \delta + 1) \to \mathcal{S}(n, \delta),$$

with the higher Tamari order $\mathcal{T}(n, \delta + 1)$ the image of this map.

Visibility in terms of cubillages: example 1



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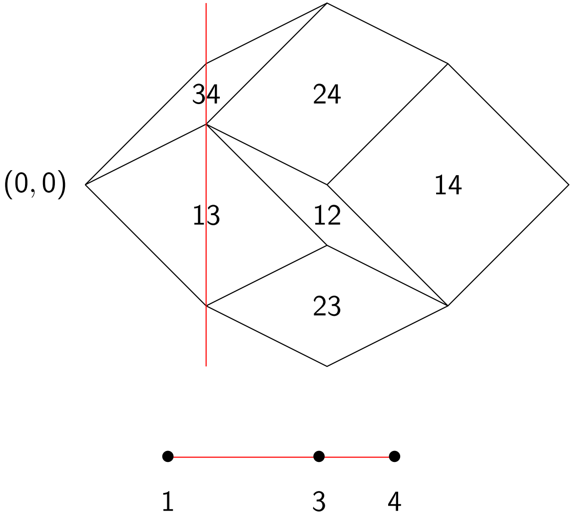
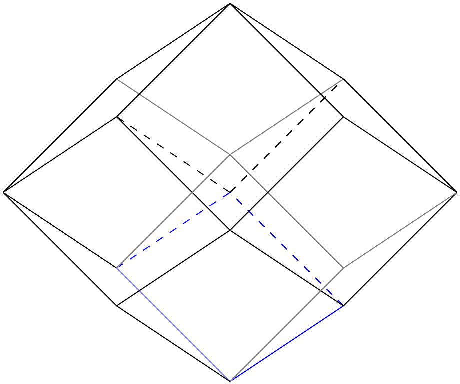


Visibility in terms of cubillages: example 2



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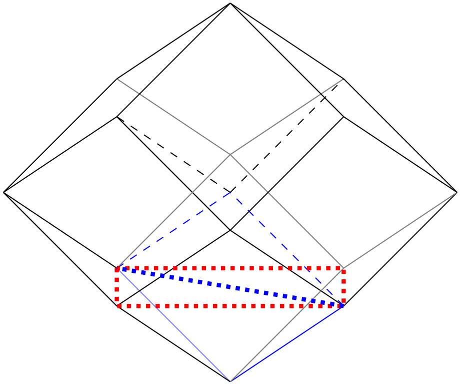

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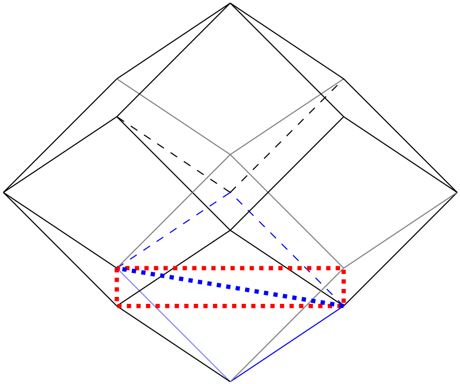

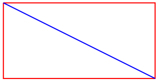

Main theorem

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Theorem ([Wil21])
$\mathcal{T}(n, \delta + 1) \cong \mathcal{S}(n, \delta)$.

Thank you very much for listening!

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