Combinatorial aspects of virtually Cohen-Macaulay sheaves

Jay Yang

joint work with Christine Berkesch, Patricia Klein, and Michael C. Loper

McMaster University

January 19, 2022

Virtual Resolutions

Definition

A *free resolution* of an S-module M is a free complex $F_\bullet$ with entries in non-negative homological degrees such that $H_0(F_\bullet) = M$ and $H_i(F_\bullet) = 0$.

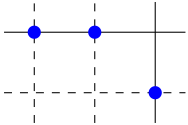
Definition (Berkesch, Erman, Smith, 2021)

Given a smooth toric variety X and an S-module M, a free complex of S-modules is a *virtual resolution* of M if the corresponding complex of vector bundles is a locally-free resolution of the sheaf $\widetilde{M}$.

Example of a Virtual Resolution

Example

$X = \mathbb{P}^1 \times \mathbb{P}^1, S = k[x_0, x_1, y_0, y_1], B = \langle x_0y_0, x_0y_1, x_1y_0, x_1y_1 \rangle$
Let us consider the variety corresponding to the three points
$\{[0,1;1,0],[1,1;1,0],[1,0;0,1]\} \subset X$



We can give two different ideals in S that give this variety

$$I = \left\langle x_1y_1, x_0^2y_0 - x_0x_1y_0 \right\rangle$$
$$J = \left\langle x_1y_1, y_0y_1, x_0^2x_1 - x_0x_1^2, x_0^2y_0 - x_0x_1y_0 \right\rangle$$

Resolution of S/I: $S^1 \leftarrow S^2 \leftarrow S^1 \leftarrow 0$.

Resolution of S/J: $S^1 \leftarrow S^4 \leftarrow S^4 \leftarrow S^1 \leftarrow 0$.

Virtually Cohen–Macaulay

Question

What bounds can we give on the length of virtual resolutions from the geometry of the corresponding variety.

Definition

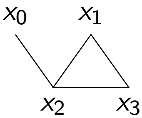
For a module M, we have that the length of the minimial free resolution ($\text{pdim}\, M$) is bounded below by $\text{codim}\, M$ with equality if M is *Cohen–Macaulay*. A module is *virtually Cohen–Macualay* if the length of the shortest virtual resolution is equal to $\text{codim}\, M$.

Stanley–Reisner Correspondance

Definition

Given a polynomial ring $S = k[x_1, \ldots, x_n]$, and a simplicial complex Δ on the variables. The *Stanley–Reisner ideal* associated to Δ, denoted I_Δ is the squarefree monomial ideal generated by monomials corresponding to the minimal non-faces of Δ.

Example

$\Delta =$ 

$$I_\Delta = \langle x_0x_1, x_0x_3, x_1x_2x_3 \rangle$$

Stanley Reisner Ideals and Reisner's Criterion

Definition

For a simplicial complex Δ, we say Δ is Cohen–Macaualay if the squarefree monomial ideal I_Δ is Cohen–Macualay ($\operatorname{pdim} I_\Delta = \operatorname{codim} I_\Delta$)

Theorem (Reisner's Criterion)

Δ is Cohen–Macualay if and only if for all $\sigma \in \Delta$ and all $i < \dim \operatorname{link}_\Delta(\sigma)$.

$$\widetilde{H}_i(\operatorname{link}_\Delta(\sigma)) = 0.$$

Virtually Cohen-Macaulay squarefree monomial ideals

Question

Which squarefree monomial ideals are virtually Cohen–Macaulay?

Theorem (Berkesch, Klein, Loper, Y. 2021)

For $X = \mathbb{P}^{n_1} \times \cdots \mathbb{P}^{n_r}$, with homogeneous coordinate ring S and if Δ is an r-dimensional equidimensional simplicial complex, then the S/I_Δ is virtually Cohen–Macaulay.

Corollary (Berkesch, Klein, Loper, Y. 2021)

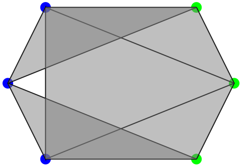
For $X = \mathbb{P}^{n_1} \times \cdots \mathbb{P}^{n_r}$, and if I is the ideal for a 1-dimensional monomial subvariety then S/I is virtually Cohen–Macaulay.

Example

$$X = \mathbb{P}^2 \times \mathbb{P}^2$$

$$I_\Delta = \langle x_0x_1x_2, y_1y_2, x_0y_0, y_1x_2, y_2x_1 \rangle$$

$$\Delta$$

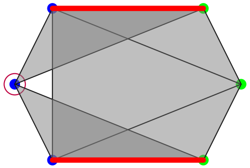


Example

$$X = \mathbb{P}^2 \times \mathbb{P}^2$$

$$I_\Delta = \langle x_0 x_1 x_2, y_1 y_2, x_0 y_0, y_1 x_2, y_2 x_1 \rangle$$

Δ

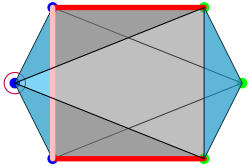


Example

$$X = \mathbb{P}^2 \times \mathbb{P}^2$$

$$I_{\Delta \cap \mathcal{B}_2} = \langle x_0 y_0, y_1 x_2, y_2 x_1 \rangle$$

$$\Delta \cup \mathcal{B}_2$$



Sketch of Proof

- Step 1: Reduce to the connected case

- Step 2: Prove either Δ or $\Delta \cup \mathcal{B}_r$ is Cohen–Macaulay. Where

$$\mathcal{B}_r := \{\sigma \mid \dim \sigma \leq r, \sigma \text{ is irrelevant}\}$$

Geometry of the Irrelevant Ideal

Goal: Show that $\Delta \cup \mathcal{B}_r$ is CM.
Tools:

- Reisner's Criterion

$$H_i(\mathrm{link}_\sigma(\Delta \cup \mathcal{B}_r)) \stackrel{?}{=} 0$$

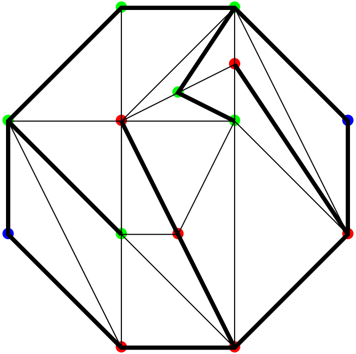
- Long Exact Sequence of a pair for $(\mathrm{link}_\sigma(\Delta'), \mathrm{link}_\sigma(\mathcal{B}_r))$ with $\Delta' = \Delta \cup \mathcal{B}_r$.

$$\cdots \to \widetilde{H}_i(\mathrm{link}_\sigma(\mathcal{B}_r)) \to \widetilde{H}_i(\mathrm{link}_\sigma(\Delta')) \to H_i(\mathrm{link}_\sigma(\Delta'), \mathrm{link}_\sigma(\mathcal{B}_r)) \to \cdots .$$

- $H_i(\mathrm{link}_\sigma(\mathcal{B}_r)) = 0$ for most σ.
- $H_i(\mathrm{link}_\sigma(\Delta'), \mathrm{link}_\sigma(\mathcal{B}_r))$ can be computed via a graph.

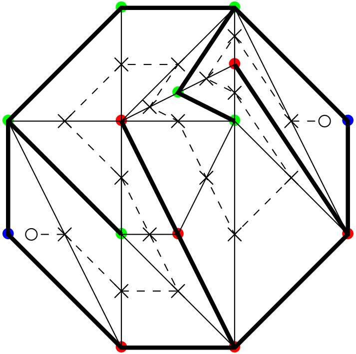
From Simplexes to Graphs

Let $X = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \mathbb{P}^{n_3}$. Color vertices from $\mathbb{P}^{n_1}$ blue, those from $\mathbb{P}^{n_2}$ red, and those from $\mathbb{P}^{n_3}$ green. Let σ be some blue vertex, then as an example suppose $\text{link}_\sigma(\Delta)$ is given by the following simplicial complex.

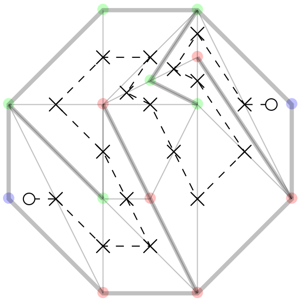


From Simplexes to Graphs

Let $X = \mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \mathbb{P}^{n_3}$. Color vertices from $\mathbb{P}^{n_1}$ blue, those from $\mathbb{P}^{n_2}$ red, and those from $\mathbb{P}^{n_3}$ green. Let σ be some blue vertex, then as an example suppose $\mathrm{link}_\sigma(\Delta)$ is given by the following simplicial complex.



$\Delta \cup \mathcal{B}_r$ is Cohen–Macaulay

G = 

Let $\widehat{G}$ be the one-point compactification of G at vertex $*$.

$$\begin{array}{ccc} C_2(\mathrm{link}_\sigma(\Delta), \mathrm{link}_\sigma(\mathcal{B}_r) \cap \mathrm{link}_\sigma(\Delta)) & \longrightarrow & C_1(\mathrm{link}_\sigma(\Delta), \mathrm{link}_\sigma(\mathcal{B}_r) \cap \mathrm{link}_\sigma(\Delta)) \\ \downarrow \cong & & \downarrow \cong \\ C_1(\widehat{G}, *) & \twoheadrightarrow & C_0(\widehat{G}, *) \end{array}$$

Future Directions

1. Virtual Reisner's Criterion
2. Virtual Shellability
3. Virtual Cellular/Simplicial Resolutions

Virtually Regular Elements

Definition

We say an element $f \in S$ is *M-regular* if $\mathrm{Ann}_M(f) := \{m \in M \mid fm = 0\} = 0$.

Definition

We say that an element $f \in S$ is *virtually M-regular* or a *virtually M-regular element* if $\mathrm{Ann}_M(f)$ is irrelevant and, $\dim M = 1 + \dim M/fM$.

Theorem (Berkesch, Klein, Loper, Y. 2021)

If f is virtually regular and M has a virtual resolution of length l, then M/fM has a virtual resolution of length $l+1$. In particular if M is vCM then M/fM is vCM or irrelevant.

Virtually Regular Elements

Example

$X = \mathbb{P}^2$, $S = k[x_0, x_1, x_2]$, $M = S/\langle x_0^2, x_0x_1\rangle$,vdim $M \geq 2$, codim $M = 1$
However, x_2 is a (virtually) regular element on M,

$$M/\langle x_2\rangle M = S/\left\langle x_0^2, x_0x_1, x_2\right\rangle$$

$$\left\langle x_0^2, x_0x_1, x_2\right\rangle = \langle x_0, x_2\rangle \cap \left\langle x_0^2, x_1, x_2\right\rangle$$

$$M/\langle x_2\rangle M^{\sim} \cong S/\langle x_0, x_2\rangle^{\sim}$$

Virtual Ext & Tor

Proposition (Berkesch, Klein, Loper, Y. 2021)

Let M and N be $\mathrm{Pic}(X)$-graded S-modules. If $F_\bullet$ is any virtual resolution of M, then

- $\mathrm{Ext}^i_S(M, N)^\sim = H_i(Hom_S(F_\bullet, N))^\sim$,
- $\mathrm{Tor}_i^S(M, N)^\sim = H_i(F_\bullet \otimes_S N)^\sim$.

Sketch of proof.

Construct a double complex for Ext/Tor and compute the appropriate spectral sequence. □

Virtual Ext & Tor

Corollary (Berkesch, Klein, Loper, Y. 2021)

If M has a virtual resolution of length ℓ then $\operatorname{Ext}^i_S(M,S)^\sim = 0$ for all $i > \ell$. In particular if M is vCM then $\operatorname{Ext}^i_S(M,S)^\sim = 0$ for $i > \operatorname{codim} M$

Remark

This provides a method (although not particularly nice) to prove that a module is **NOT** vCM.

Mapping Cone Trick

Let M be an S-module with a free resolution $F_\bullet$ of length t and $\operatorname{Ext}^t(M,S)^\sim = 0$. Then consider $G_\bullet$ a free resolution of $\operatorname{Ext}^t(M,S)$. We get a map $\alpha_t^* : F_t^* \to G_0$

$$\begin{array}{ccccccccccccc}
\cdots & \longrightarrow & 0 & \longrightarrow & F_0^* & \xrightarrow{\varphi_1^*} & \cdots & \xrightarrow{\varphi_{t-1}^*} & F_{t-1}^* & \xrightarrow{\varphi_t^*} & F_t^* & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \alpha_0^*} & & \downarrow{\scriptstyle \alpha_1^*} & & & & \downarrow{\scriptstyle \alpha_{t-1}^*} & & \downarrow{\scriptstyle \alpha_t^*} & & \\
\cdots & \longrightarrow & G_{t+1} & \xrightarrow{\psi_{t+1}} & G_t & \xrightarrow{\psi_t} & \cdots & \xrightarrow{\psi_1} & G_1 & \xrightarrow{\psi_0=\varphi_t^*} & G_0 & \longrightarrow & 0.
\end{array}$$

Mapping Cone Trick

Now dualize to get the following diagram:

$$
\begin{array}{ccccccccccccc}
\cdots & \longleftarrow & 0 & \longleftarrow & F_0 & \xleftarrow[\varphi_2]{} & \cdots & \xleftarrow[\varphi_{t-1}]{} & F_{t-1} & \xleftarrow[\varphi_t]{} & F_t & \longleftarrow & 0 \\
 & & \Big\uparrow{\scriptstyle \alpha_{-1}} & & \Big\uparrow{\scriptstyle \alpha_0} & & & & \Big\uparrow{\scriptstyle \alpha_{t-1}} & & \Big\uparrow{\scriptstyle \alpha_t} & & \\
\cdots & \xleftarrow[\psi_{t+1}]{} & G_{t+1}^* & \xleftarrow[\psi_t]{} & G_t^* & \xleftarrow[\psi_{t-1}]{} & \cdots & \xleftarrow[\psi_1]{} & G_1^* & \xleftarrow[\psi_0=\varphi_t]{} & G_0^* & \longleftarrow & 0.
\end{array}
$$

Mapping Cone Trick

Proposition (Berkesch, Klein, Loper, Y. 2021)

If $G_{t+2} = 0$, with α as before cone(α) *is a virtual resolution of M.*

$$0 \xleftarrow{\partial_0} \begin{matrix} F_0 \\ \oplus \\ G_{t+1}^* \end{matrix} \xleftarrow{\partial_1} \begin{matrix} F_1 \\ \oplus \\ G_t^* \end{matrix} \xleftarrow{\partial_2} \cdots \xleftarrow{\partial_{t-1}} \begin{matrix} F_{t-1} \\ \oplus \\ G_2^* \end{matrix} \xleftarrow{\partial_t} \begin{matrix} F_t \\ \oplus \\ G_1^* \end{matrix} \xleftarrow{\partial_{t+1}} \begin{matrix} 0 \\ \oplus \\ G_0^* \end{matrix} \leftarrow 0.$$

Mapping Cone Trick

Proposition (Berkesch, Klein, Loper, Y. 2021)

If $G_{t+2} = 0$*, with* α *as before* $\mathrm{cone}(\alpha)$ *is a virtual resolution of* M*.*

$$0 \xleftarrow{\partial_0} \begin{matrix} F_0 \\ \oplus \\ G^*_{t+1} \end{matrix} \xleftarrow{\partial_1} \begin{matrix} F_1 \\ \oplus \\ G^*_t \end{matrix} \xleftarrow{\partial_2} \cdots \xleftarrow{\partial_{t-2}} \begin{matrix} F_{t-2} \\ \oplus \\ G^*_3 \end{matrix} \xleftarrow{\partial_{t-1}} G^*_2 \leftarrow 0$$

Mapping Cone Example

Let $X = \mathbb{P}^3$ and $I = \langle x_0x_2, x_0x_3, x_1x_2, x_1x_3 \rangle$

$$\mathrm{Ext}^3(S/I, S) = S/B = S/\langle x_0, x_1, x_2, x_3 \rangle$$

$$F_\bullet : S^1 \leftarrow S^4 \leftarrow S^4 \leftarrow S^1$$
$$G_\bullet : S^1 \leftarrow S^4 \leftarrow S^6 \leftarrow S^4 \leftarrow S^1$$

Mapping Cone Example (cont.)

$$F_\bullet : S^1 \leftarrow S^4 \leftarrow S^4 \leftarrow S^1$$

$$G_\bullet : S^1 \leftarrow S^4 \leftarrow S^6 \leftarrow S^4 \leftarrow S^1$$

$$\begin{array}{ccccccccccccc}
 & & 0 & \longrightarrow & S^1 & \longrightarrow & S^4 & \longrightarrow & S^4 & \longrightarrow & S^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & S^1 & \longrightarrow & S^4 & \longrightarrow & S^6 & \longrightarrow & S^4 & \longrightarrow & S^1 & \longrightarrow & 0.
\end{array}$$

Mapping Cone Example (cont.)

$$\begin{array}{ccccccccccccc}
 & & 0 & \longleftarrow & S^1 & \longleftarrow & S^4 & \longleftarrow & S^4 & \longleftarrow & S^1 & \longleftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & S^1 & \longleftarrow & S^4 & \longleftarrow & S^6 & \longleftarrow & S^4 & \longleftarrow & S^1 & \longleftarrow & 0.
\end{array}$$

Mapping Cone Example (cont.)

$$\begin{array}{ccccccccccccc}
 & & 0 & \longleftarrow & S^1 & \longleftarrow & S^4 & \longleftarrow & S^4 & \longleftarrow & S^1 & \longleftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & S^1 & \longleftarrow & S^4 & \longleftarrow & S^6 & \longleftarrow & S^4 & \longleftarrow & S^1 & \longleftarrow & 0.
\end{array}$$

$$\text{cone}(\alpha) : S^2 \leftarrow S^8 \leftarrow S^{10} \leftarrow S^5 \leftarrow S^1 \leftarrow 0$$

$$F'_\bullet : S^2 \xleftarrow{d_0} S^4 \leftarrow S^2 \leftarrow 0$$

Mapping Cone Example (cont.)

$$\begin{array}{ccccccccccccc}
 & & 0 & \longleftarrow & S^1 & \longleftarrow & S^4 & \longleftarrow & S^4 & \longleftarrow & S^1 & \longleftarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longleftarrow & S^1 & \longleftarrow & S^4 & \longleftarrow & S^6 & \longleftarrow & S^4 & \longleftarrow & S^1 & \longleftarrow & 0.
\end{array}$$

$$\mathrm{cone}(\alpha) : S^2 \leftarrow S^8 \leftarrow S^{10} \leftarrow S^5 \leftarrow S^1 \leftarrow 0$$

$$F'_\bullet : S^2 \xleftarrow{d_0} S^4 \leftarrow S^2 \leftarrow 0$$

$$d_0 = \begin{bmatrix} 0 & 0 & x_3 & x_2 \\ -x_0 & -x_1 & -x_3 & -x_2 \end{bmatrix}$$