The Combinatorial PT-DT Correspondence

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Summary:

We prove the identity

$$V(\mu_1, \mu_2, \mu_3) = \prod_{n \geq 1} \left(\frac{1}{1-q^n}\right)^n W(\mu_1, \mu_2, \mu_3)$$

- μ_1, μ_2, μ_3 are arbitrary integer partitions (parameters of both V, W)
- V q-counts plane-partition-like objects
- W q-counts a more exotic type of "box configurations" related to the double dimer model.

Summary (Cont'd)

This identity was conjectured in geometry: V, W are "topological vertices" (local contribution to curve counts in a toric Calabi-Yau 3-fold).

$V \rightsquigarrow$ Donaldson-Thomas (DT) vertex (MNOP 2004)

$W \rightsquigarrow$ Pandharipande-Thomas (PT) vertex ($\mu_3 = \phi$: PT 2009; General case: JWY 2021)

Summary (cont'd):

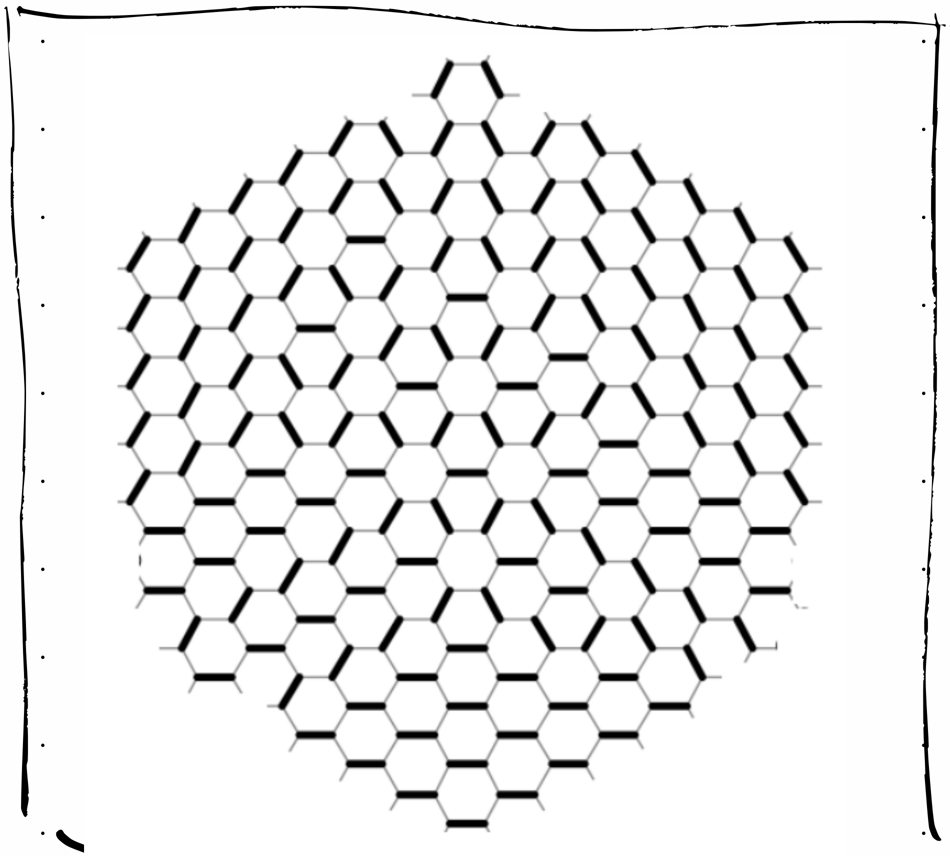
We don't do <u>any</u> geometry – "just" enumeration.

- we relate the geometer's "box counting" problems to the $\left\{\begin{matrix}\text{single-}\\ \text{double-}\end{matrix}\right\}$ dimer model;
- we show that both U, W satisfy the same recurrence (coming from "condensation" in the 2 models)

Outline

- $\left.\begin{array}{l}\text{Dimer}\\ \text{Double-dimer}\end{array}\right\}$ models
- $\left.\begin{array}{l}\text{DT}\\ \text{PT}\end{array}\right\}$ Box configurations
- Condensation recurrence.

Dimer Model



μ_1, μ_2, μ_3 encode boundary conditions near the corners (via their Maya diagrams)

Kasteleyn (1961) and many others.

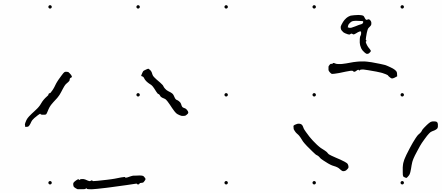
Dimer model.

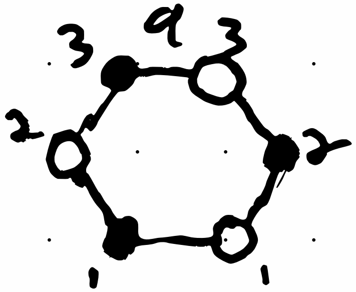
Kasteleyn matrix K: white-to-black adjacency matrix with signs and weights

For us, all signs are $+1$

all weights are q^m, $m \in \mathbb{N}$.

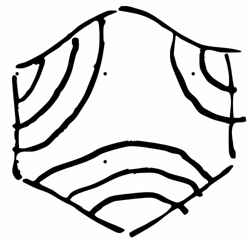
$$|\operatorname{Det} K| = \sum_{\substack{M\ \cancel{\text{perfect}}\\ \text{matching}}} \prod_{e \in M} w(e).$$

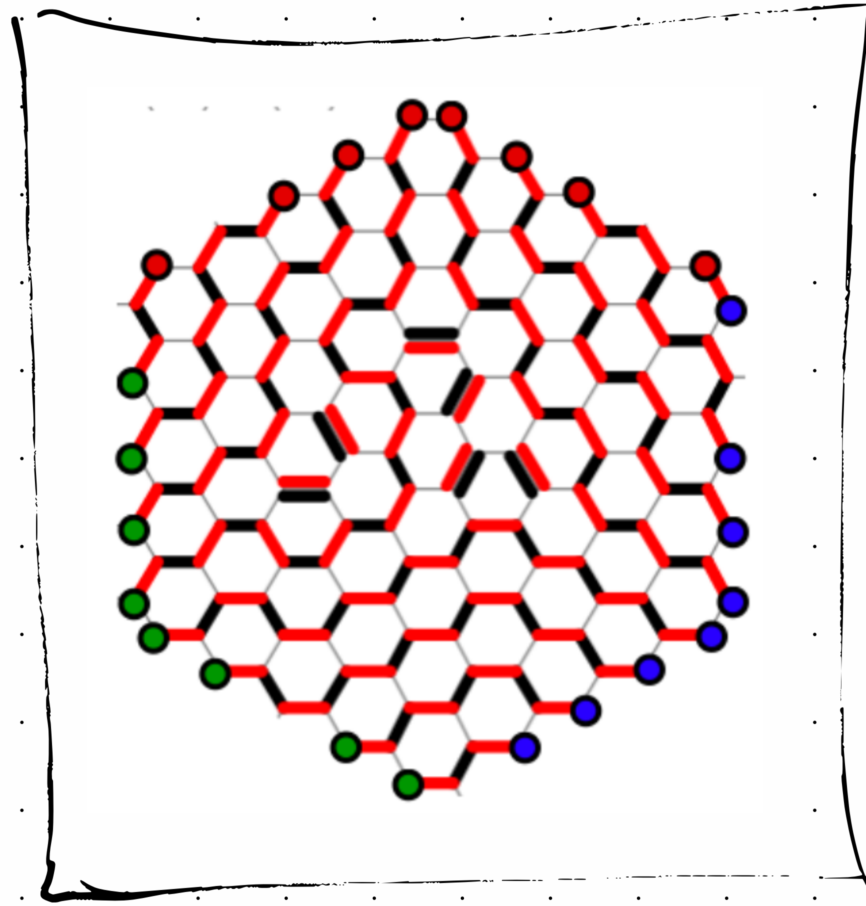




$$K = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & q \end{bmatrix} \quad \det K = -(1 + q)$$

Tripartite Double-Dimer Model

- some boundary vertices identified as <u>nodes</u>.
- Connectivity is 
- μ_1, μ_2, μ_3 encode what vertices <u>aren't</u> nodes.



[studied by Jenne, 2020; FPSAC 2020
Kenyon-Wilson 2008)

Kenyon-Wilson, Jenne both give matrices whose determinants count such configurations.

I'll call it

However, for Jenne's matrix, J, minors also count configs, with some nodes turned off.

Rows: black nodes
cols: white nodes.

Box counting in DT, PT theory.

We'll now describe "box configurations" (basically some special $\mathbb{C}[x_1, x_2, x_3]$-modules) that we're counting for the geometric theories.

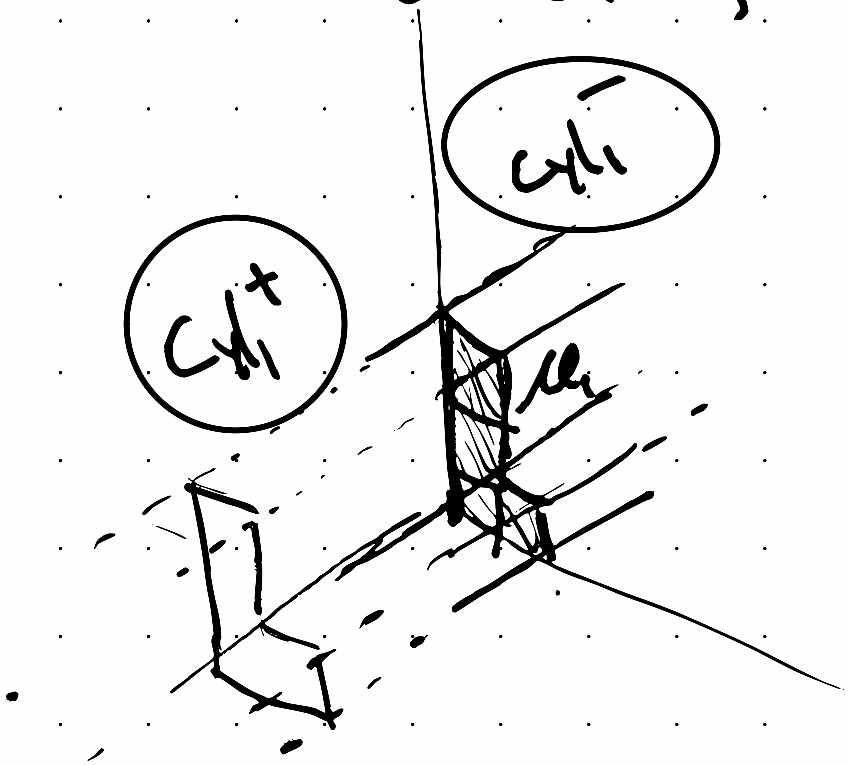
- subsets of $\mathbb{Z}^3$ where each point is drawn as a unit cube.

The 3 Cylinders

μ_1, μ_2, μ_3 Fixed Young dia.

Cyl_1 = cylinder with cross-section μ_1 extending out the x axis, in both directions

Similarly Cyl_2 on y axis
Cyl_3 on z axis.

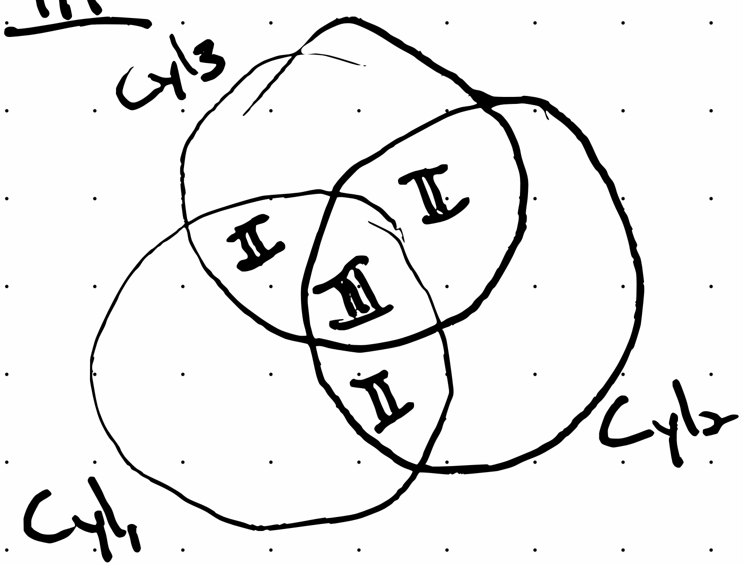


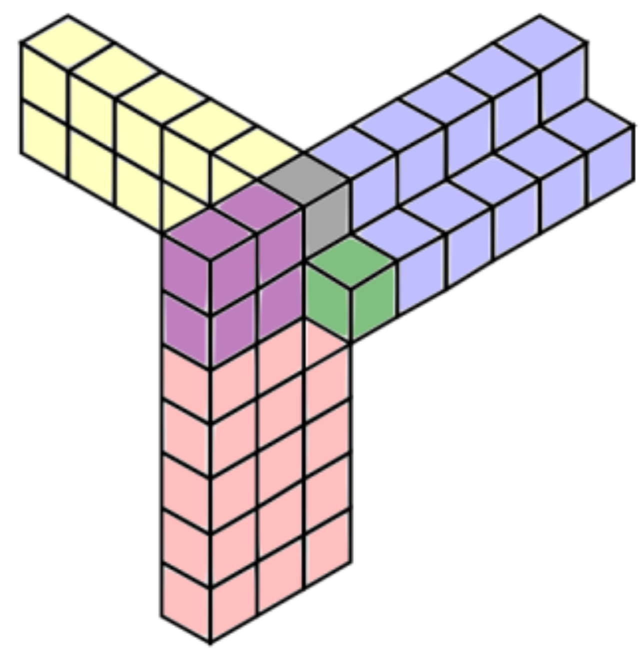
Intersections of the cylinders

All 3 cylinders meet, in a complicated way, in 1st octant.

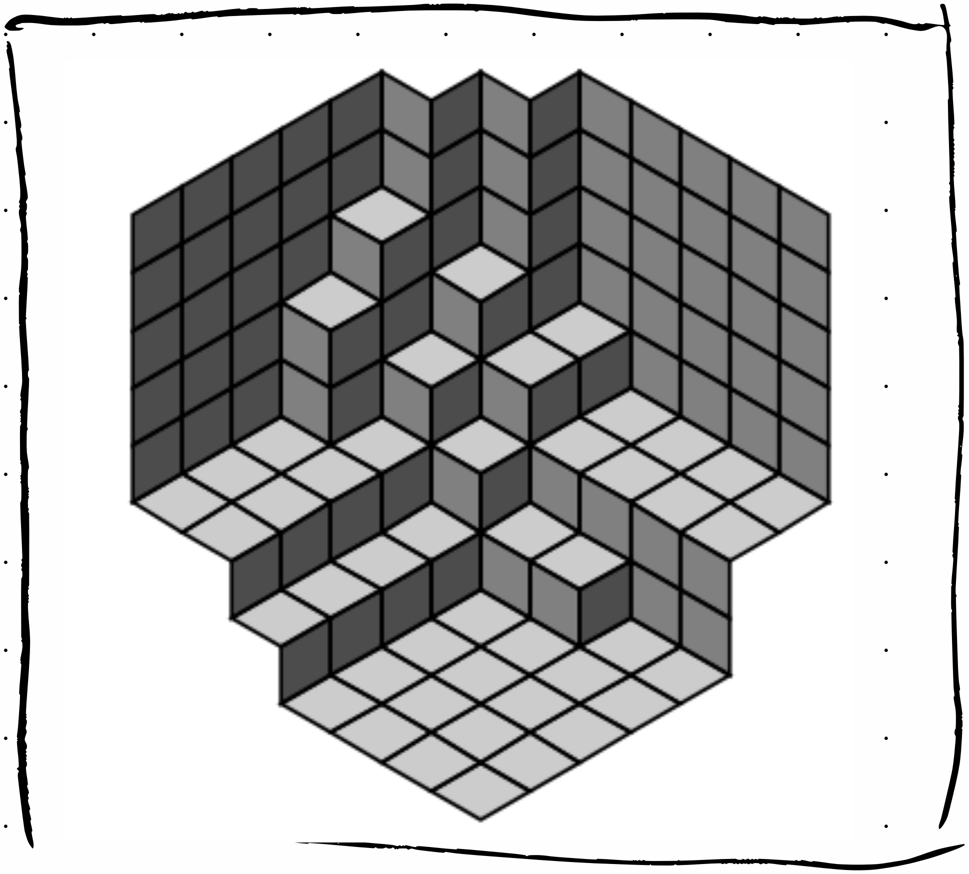
II = 2-fold intersections

III = 3-fold intersections.





DT box configs – "Plane partitions asymptotic to (μ_1, μ_2, μ_3)"

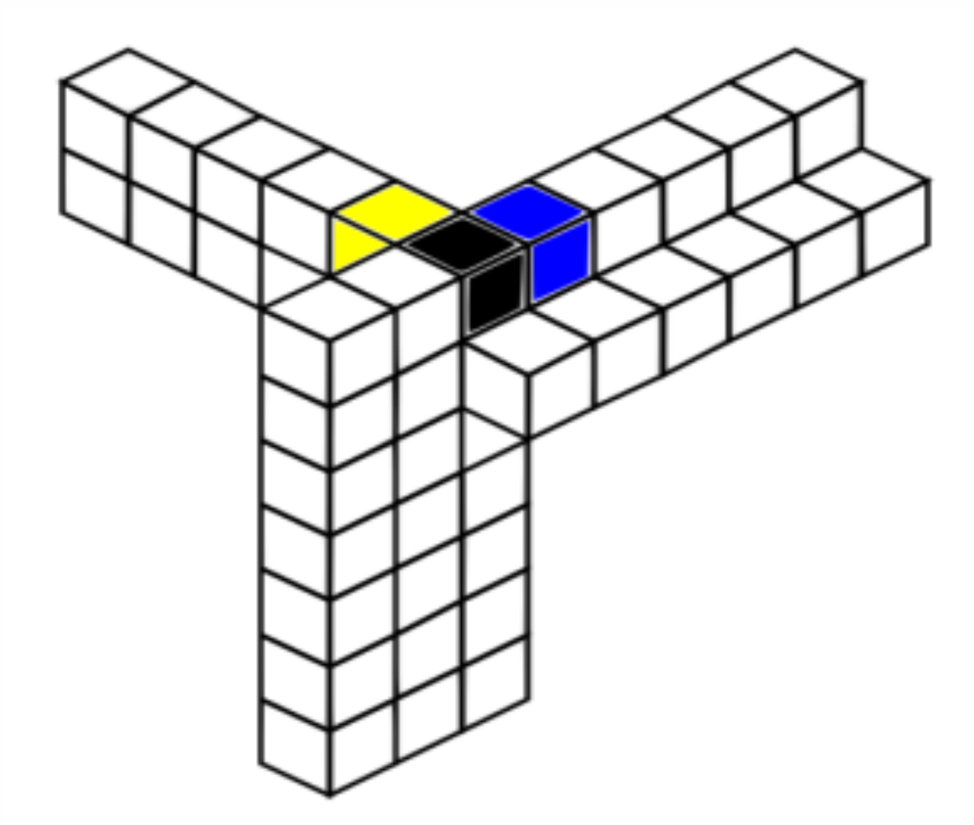


- Schur function expansion: ORV 2003 (which we do not use)

Stack boxes on top of $Cyl_1^+ \cup Cyl_2^+ \cup Cyl_3^+$

Ex If $\mu_1 = \mu_2 = \mu_3 = \phi$: these are plane partitions.

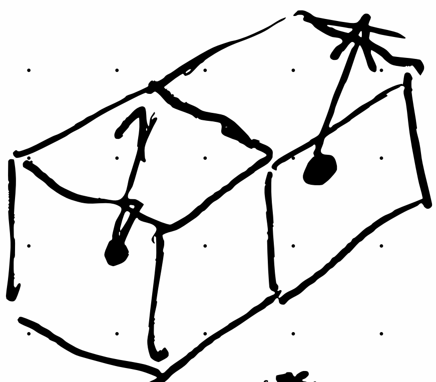
PT box Configs



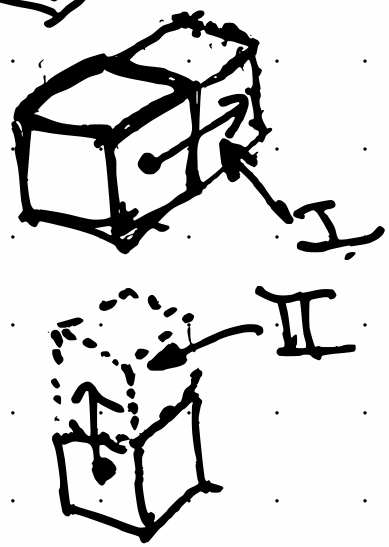
Boxes stacked inside $\mathrm{II} \cup \mathrm{III} \cup \bigcup_i \mathrm{Cyl}_i^-$

- Gravity pulls in $(1,1,1)$ direction
- Boxes in II either labels or have multiplicity 2.

PT box configs – labels (sketch)



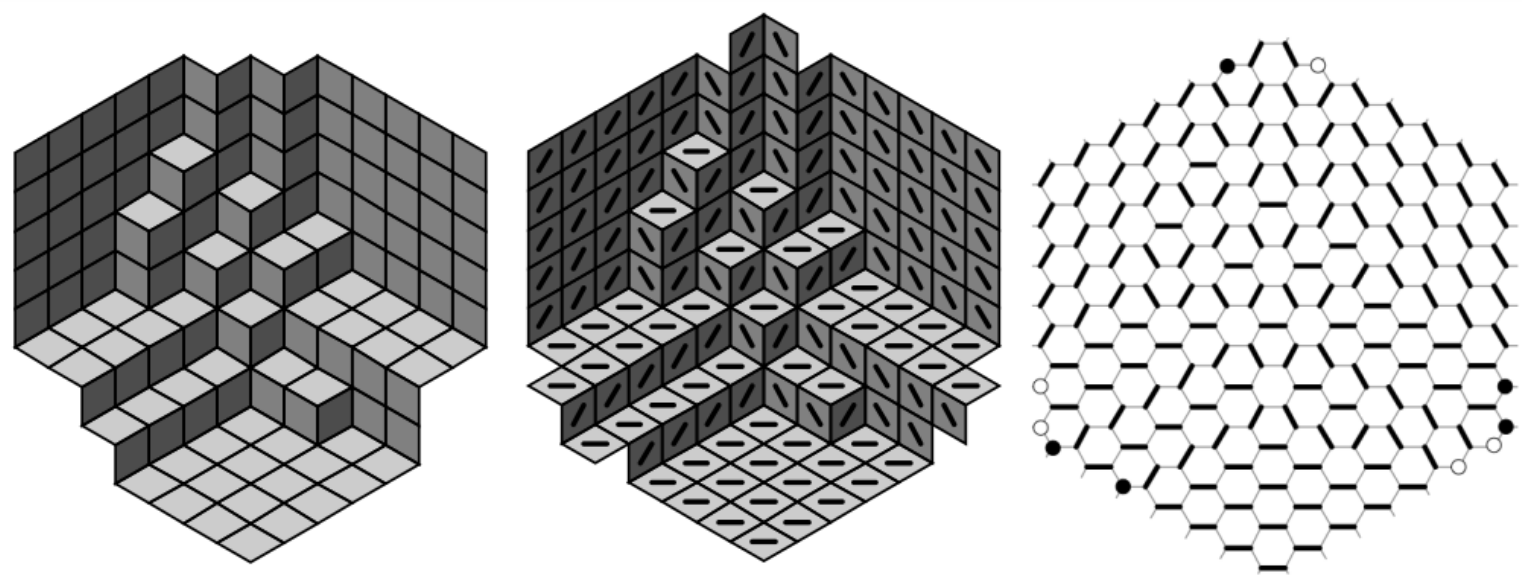
- Labels of adjacent boxes in III must be the same.



- Labels are constrained by presence or absence of adjacent non-III boxes

- If a label is "overconstrained" the box must appear with multiplicity 2.

Relating the dimers to box counts
– the "Folklore" correspondences between matchings, tilings, plane partitions.



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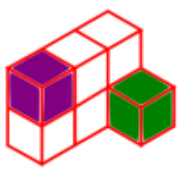

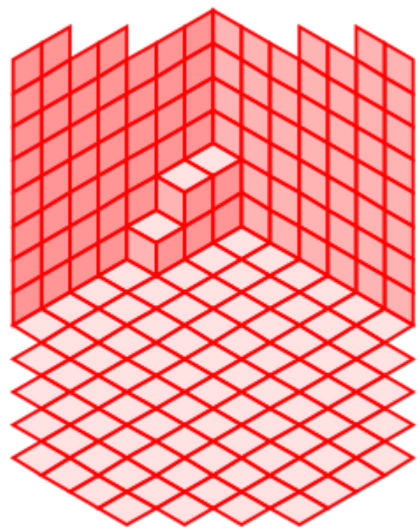

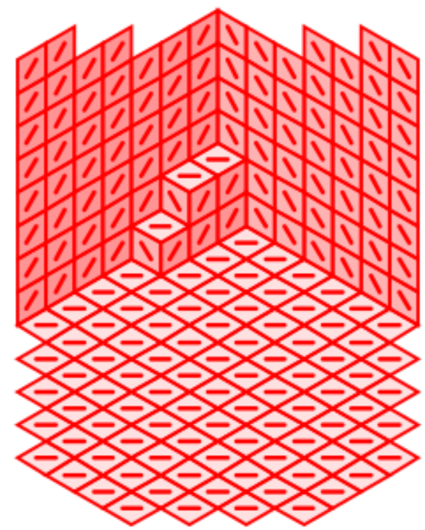

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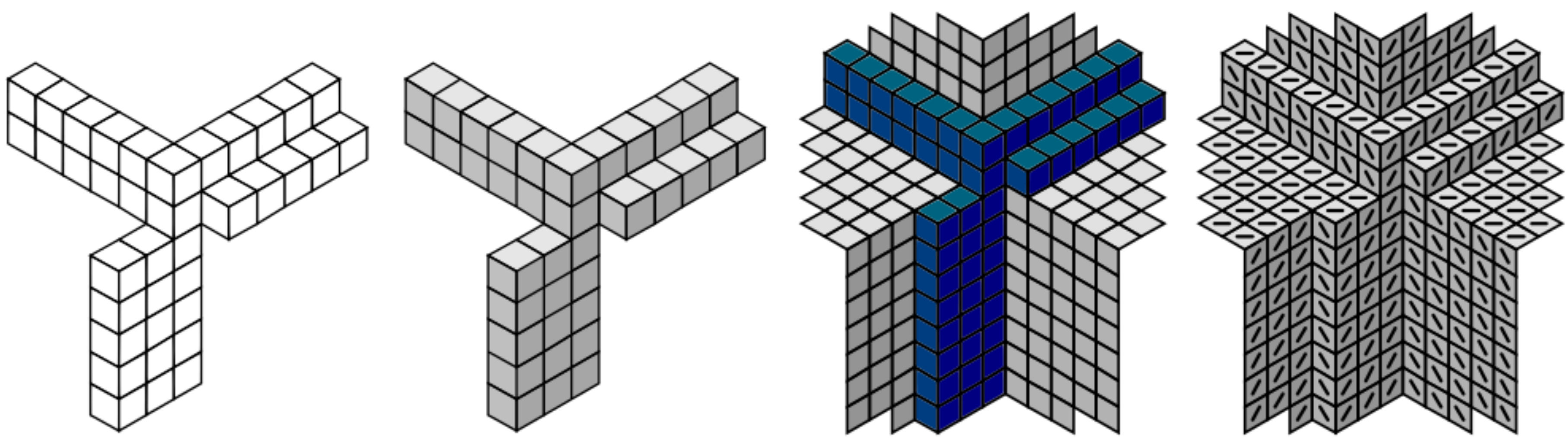

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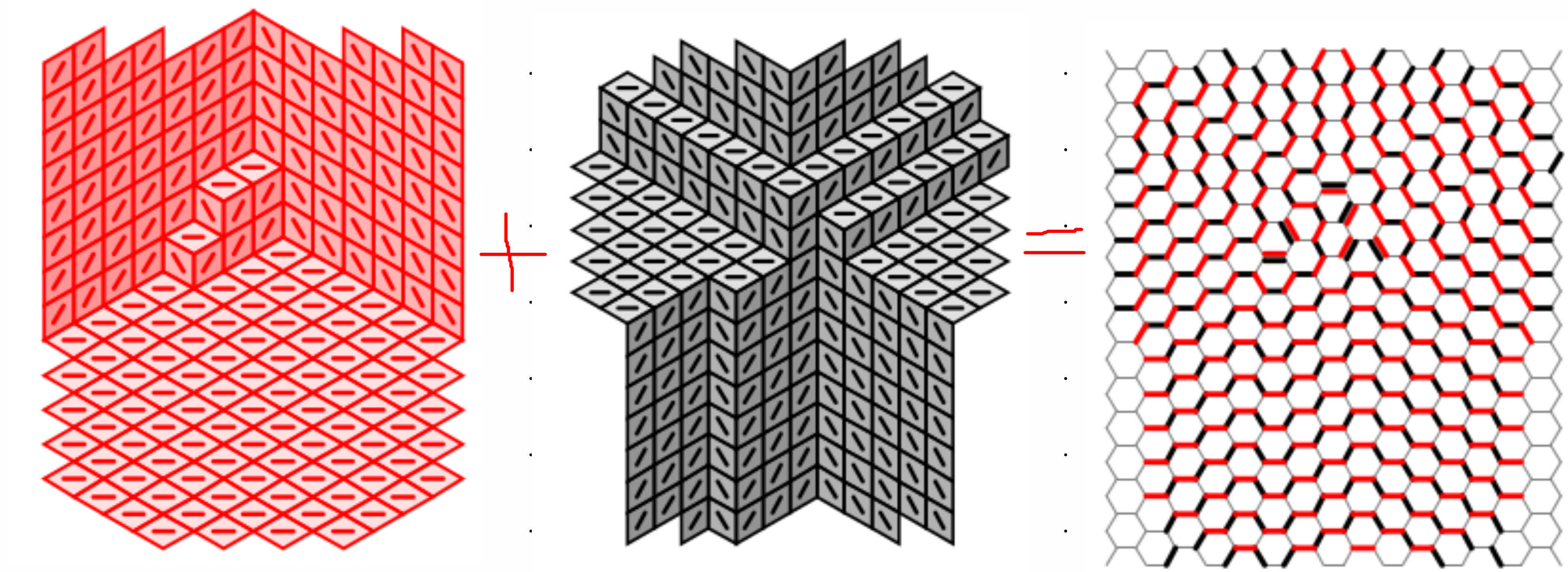


- Labelling rules $\longleftrightarrow$ tripartiteness of double-dimers.

The generating functions:

$$V(\mu_1,\mu_2,\mu_3) = \sum_{\substack{\lambda \text{ asymp.} \\ \text{to } \mu_1,\mu_2,\mu_3}} q^{|\lambda|}$$

where

$$|\lambda| = \#\{\text{boxes in } \lambda \setminus \text{min. config}\} - |\mathrm{II}| - 2|\mathrm{III}|$$

$$\text{min config} = \mathrm{cyl}_1^+ \cup \mathrm{cyl}_2^+ \cup \mathrm{cyl}_3^+$$

The generating functions:

$$W = \sum_{\substack{\lambda \text{ PT box} \\ \text{config}}} \chi_{top}(\lambda)\, q^{|\lambda|}$$

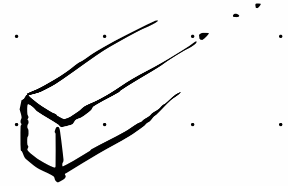
$\chi_{top}(\lambda)$: topological Euler characteristic of moduli space of labellings (basically $2^{\#\{\text{free choices of label}\}}$)

$$|\lambda| = \#\left\{\substack{\text{boxes in } \lambda \text{ up to} \\ \text{multiplicity}}\right\} - |\mathrm{II}| - 2|\mathrm{III}|$$

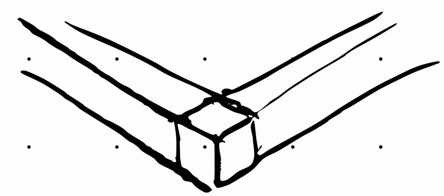
Examples

(everything is trivial)

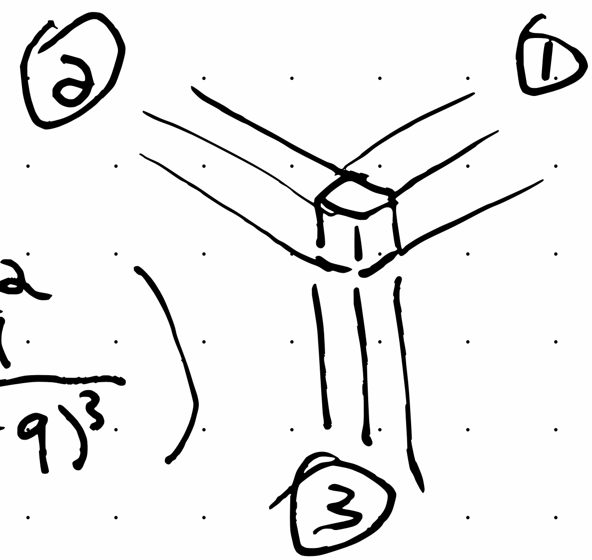
$$W_{\emptyset\emptyset\emptyset} = 1$$



$$W_{\square\emptyset\emptyset} = \frac{1}{1-q}$$

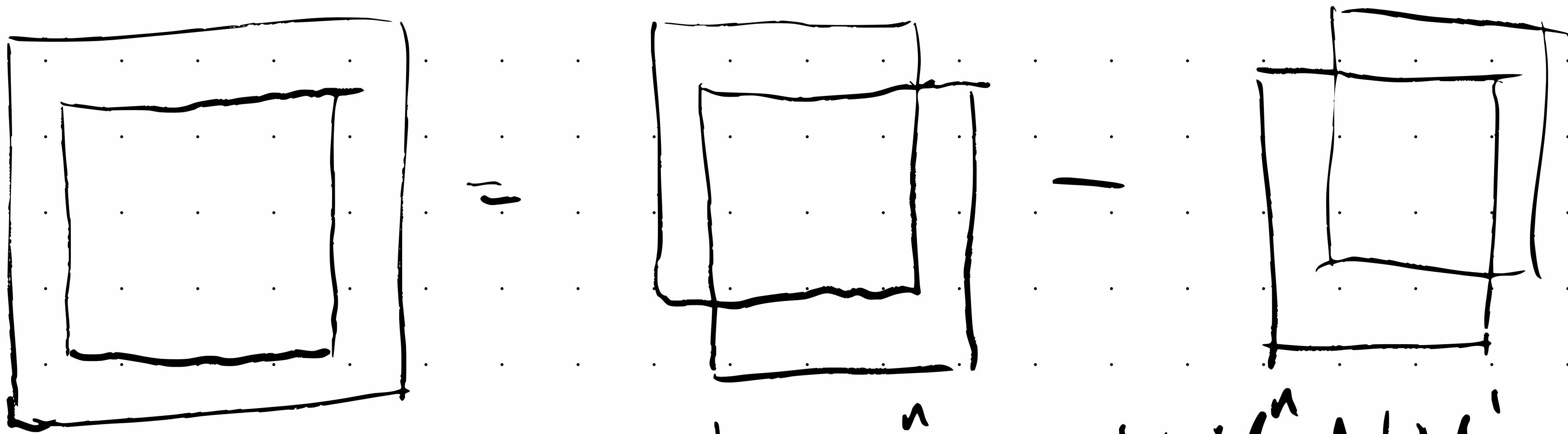


$$W_{\square\square\emptyset} = q^{-1}\left(1 + \frac{q}{(1-q)^3}\right)$$



$$W_{\square\square\square} = q^{-2}\left(1 + 2q + 3\frac{q^2}{(1-q)} + \frac{q^2}{(1-q)^3}\right)$$

The recurrence: Condensation.



$$\det M \,\det M_{1n}^{1n} = \det M_1^1 \det M_n^n - \det M_1^n \det M_n^1$$

M_i^j means: M with row i, col j deleted.

Apply this recurrence to K, J.

Kasteleyn Matrix — Jenne Matrix

In both cases, when applied to the specific matrices I've described, one obtains the recurrence:

$$\boxed{\begin{aligned} &q^K V(\mu_1^{rc},\mu_2^{rc},\mu_3)\, V(\mu_1,\mu_2,\mu_3) \\ &\quad = q^K V(\mu_1^{rc},\mu_2,\mu_3)\, V(\mu_1,\mu_2^{rc},\mu_3) \\ &\qquad + V(\mu_1^{r},\mu_2^{c},\mu_3)\, V(\mu_1^{c},\mu_2^{r},\mu_3) \end{aligned}}$$

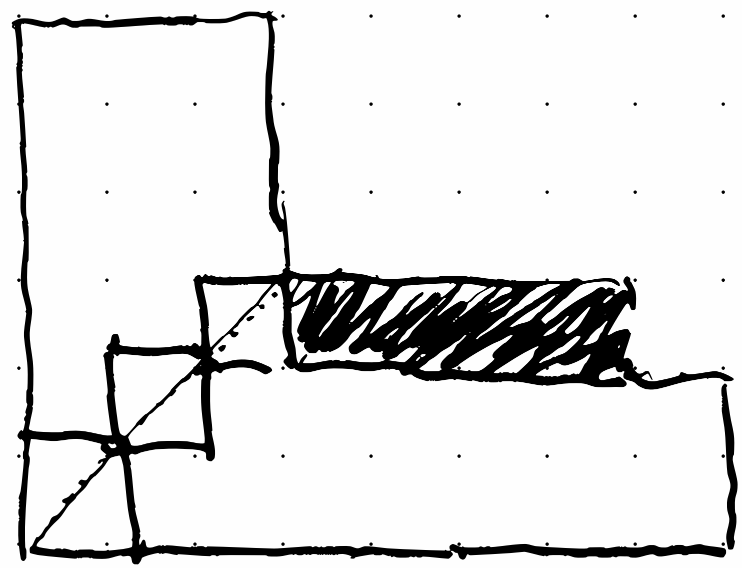
K: next slide...

and similarly for W. $\mu_i^r, \mu_i^c, \mu_i^{rc}$ mean to modify a specific row, column of μ_i.

Here $K = K(\mu_1, \mu_2)$ (not μ_3 dependent)

$$= 1 + (\mu_1)_{d(\mu_1)} - d(\mu_1)$$
$$+ (\mu_2')_{d(\mu_2)} - d(\mu_2)$$

$d(\lambda)$: diag. of λ.



Ex If $\mu_1 = \mu_2 = \mu_3 = \square$,

$$qV(\square\square\square)V(\phi\phi\square) = qV(\square\phi\square)V(\phi\square\square) + V(\phi\phi\square)^2$$

or $$V(\square\square\square) = q\frac{V(\square\square\phi)^2}{V(\square\phi\phi)} + V(\phi\phi\square).$$

(Note: this lets you compute $V(\square\square\square)$ in terms of 1- and 2-leg vertices)

Same for ω.

Note: it's clear that you'll get a recurrence for both the DT and the PT problems. It is not clear why the recurrence is the same.

Future work: other relations between the single- and double-diner models and associated "box counting" problems.

Thank You!