Perfect models and Gelfand W-graphs

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Iwahori-Hecke algebras

- Consider a finite Coxeter system (W, S) with length map $\ell : W \to \mathbb{N}$.
- The *Iwahori-Hecke algebra* of W is the module

$$\mathcal{H}(W) = \mathbb{Q}[q, q^{-1}]\text{-span}\{H_w : w \in W\}$$

with multiplication satisfying, for $s \in S$ and $w \in W$:

$$H_s H_w = \begin{cases} H_{sw} & \text{if } \ell(sw) > \ell(w) \\ H_{sw} + (q - q^{-1})H_w & \text{if } \ell(sw) < \ell(w). \end{cases}$$

When $q = 1$ this recovers the group algebra $\mathbb{Q}W$.
- A *Gelfand model* for $\mathcal{H}(W)$ is a representation that decomposes into a multiplicity-free direct sum of all irreducible $\mathcal{H}(W)$-representations.

Gelfand models for Hecke algebras of symmetric groups

Adin, Postnikov, Roichman (2008): $\mathcal{H}(S_n)$ has a "combinatorial" Gelfand model.

An integer i is a *visible descent* of a permutation z if $z(i+1) < \min\{i, z(i)\}$.

Theorem (Adin–Postnikov–Roichman (2008))

Let $\mathcal{K}(S_n) = \{$ *fixed-point-free involutions* $z \in S_{2n}$ *with no visible descents* $> n$ $\}$. *Let* $u_+ = q$ *and* $u_- = -q^{-1}$. *There is a unique* $\mathcal{H}(S_n)$*-module structure on*

$$\mathcal{M}^{\pm}(S_n) = \mathbb{Q}[q, q^{-1}]\text{-span}\{M_z : z \in \mathcal{K}(S_n)\}$$

such that if $s \in S := \{s_i = (i, i+1) : i \in [n-1]\}$ *and* $z \in \mathcal{K}(S_n)$ *then*

$$H_s M_z = \begin{cases} u_{\pm} M_z & \text{if } s \in \mathrm{Des}^{=}(z) := \{s \in S : zsz = s\} \\ u_{\mp} M_z & \text{if } s \in \mathrm{Asc}^{=}(z) := \{s \in S : zsz \in \{s_i : i > n\}\} \\ M_{szs} + (q - q^{-1}) M_z & \text{if } s \in \mathrm{Des}^{<}(z) := \{ \text{ remaining descents of } z \} \\ M_{szs} & \text{if } s \in \mathrm{Asc}^{<}(z) := \{ \text{ remaining ascents of } z \}. \end{cases}$$

Both $\mathcal{M}^+(S_n)$ *and* $\mathcal{M}^-(S_n)$ *are Gelfand models for* $\mathcal{H}(S_n)$.

Gelfand models for Hecke algebras of classical Weyl groups

The other classical Weyl groups besides S_n are

$$W_n^{\mathrm{BC}} = \{ \text{ permutations } w \text{ of } \{\pm 1, \pm 2, \dots, \pm n\} \text{ with } w(-i) = -w(i) \text{ } \},$$
$$W_n^{\mathrm{D}} = \{ \text{ permutations } w \in W_n^{\mathrm{BC}} \text{ with even number of sign changes } \}.$$

Theorem (Marberg–Z. (2020))

The formulas on the previous slide also define $\mathcal{H}(W)$-module structures on

$$\mathcal{M}^{\pm}(W) = \mathbb{Q}[q, q^{-1}]\text{-span}\{M_z : z \in \mathcal{K}(W)\}$$

for each classical Weyl group $W \in \{S_n, W_n^{\mathrm{BC}}, W_n^{\mathrm{D}}\}$, for a certain set $\mathcal{K}(W)$. Both $\mathcal{M}^{+}(W)$ and $\mathcal{M}^{-}(W)$ are Gelfand models unless $W = W_n^{\mathrm{D}}$ with n even.

To be precise, say $w \in W_n^{\mathrm{BC}}$ is *abs.-fixed-point-free* if $|w(i)| \neq i$ for all i. Then:

$$\mathcal{K}(W_n^{\mathrm{BC}}) = \{ \text{ abs.-f.p.f. involutions } z \in W_{2n}^{\mathrm{BC}} \text{ with no visible descents } > n\},$$
$$\mathcal{K}(W_n^{\mathrm{D}}) = \{ \text{ } z \in \mathcal{K}(W_n^{\mathrm{BC}}) \text{ for which } |\{i \in [n] : z(i) < -i\}| \text{ is even } \}.$$

Canonical bases

- There is a unique ring involution $\mathcal{H}(W) \to \mathcal{H}(W)$, written $H \mapsto \overline{H}$ and called the *bar operator*, such that $\overline{q} = q^{-1}$ and $\overline{H_s} = H_s^{-1}$ for all $s \in S$.
- The *Kazhdan-Lusztig basis* consists of the unique elements $\underline{H}_w \in \mathcal{H}(W)$ for $w \in W$ satisfying $\overline{\underline{H}_w} = \underline{H}_w \in H_w + \sum_{\ell(y)<\ell(w)} q^{-1}\mathbb{Z}[q^{-1}]H_y$.

Theorem (Marberg–Z. (2020))

Let $W \in \{S_n, W_n^{\mathrm{BC}}, W_n^{\mathrm{D}}\}$. There is a unique $\mathbb{Z}$-linear map $\mathcal{M}^{\pm}(W) \to \mathcal{M}^{\pm}(W)$, written $M \mapsto \overline{M}$ and called the bar operator, such that

$$\overline{HM} = \overline{H} \cdot \overline{M} \text{ for } H \in \mathcal{H}(W),\ M \in \mathcal{M}^{\pm}(W) \quad \text{and} \quad \overline{M_z} = M_z \text{ if } \mathrm{Des}^{<}(z) = \varnothing.$$

Furthermore, $\mathcal{M}^{\pm}(W)$ has a unique "canonical" basis $\{C_z : z \in \mathcal{K}(W)\}$ with

$$\overline{C_z} = C_z + \sum_{\substack{y \in \mathcal{K}(W) \\ \ell(y)<\ell(w)}} q^{-1}\mathbb{Z}[q^{-1}]M_y.$$

Gelfand W-graphs

Continue to assume $W \in \{S_n, W_n^{\mathrm{BC}}, W_n^{\mathrm{D}}\}$.

Let $\mathrm{Asc}^+(z) = \mathrm{Asc}^<(z) \sqcup \mathrm{Asc}^=(z)$ and $\mathrm{Asc}^-(z) = \mathrm{Asc}^<(z) \sqcup \mathrm{Des}^=(z)$.

Theorem

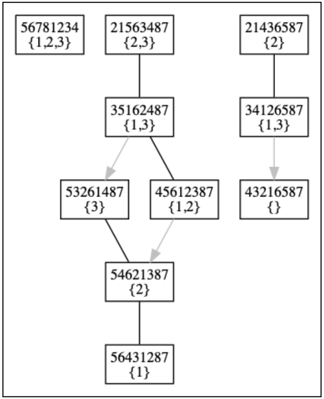
There is a $\mathbb{Z}$-weighted directed graph $\Gamma^\pm(W)$ on $\mathcal{K}(W)$ such that if $s \in S$ then

$$H_s C_z = \begin{cases} qC_z & \text{if } s \notin \mathrm{Asc}^\pm(z) \\ -q^{-1}C_z + \displaystyle\sum_{y \in \mathcal{K}(W),\ s \notin \mathrm{Asc}^\pm(y)} \omega(y,z) C_y & \text{if } s \in \mathrm{Asc}^\pm(z) \end{cases}$$

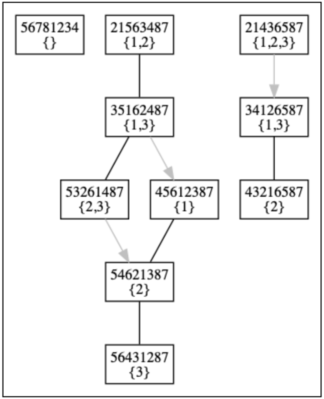
holds in $\mathcal{M}^\pm(W)$, where $\omega(y,z) \in \mathbb{Z}$ is the edge weight of $y \to z$ in $\Gamma^\pm(W)$.

- Both $\Gamma^+(W)$ and $\Gamma^-(W)$ are examples of *(quasi-admissible) W-graphs*.
- A W-graph is a certain directed graph encoding a representation of $\mathcal{H}(W)$.
- Both $\Gamma^+(W)$ and $\Gamma^-(W)$ are *Gelfand W-graphs* (outside type D in even rank): their associated $\mathcal{H}(W)$-representations are Gelfand models.

Gelfand W-graphs for S_3

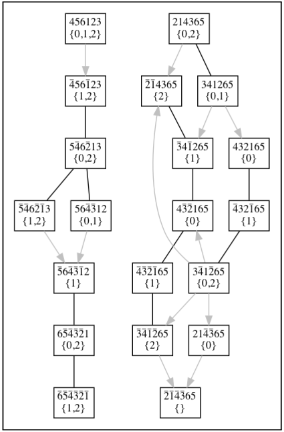


$\Gamma^+(S_3)$

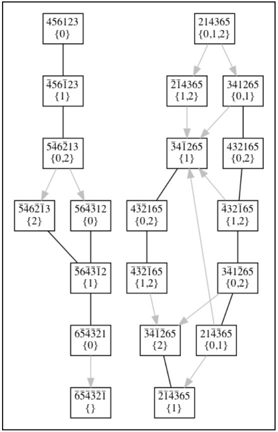


$\Gamma^-(S_3)$

Gelfand W-graphs for W_3^{BC}

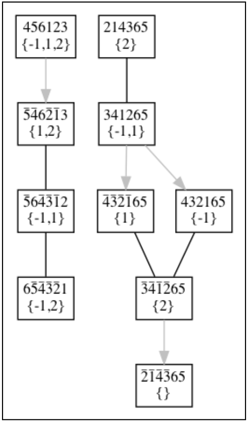


$\Gamma^+(W_3^{BC})$

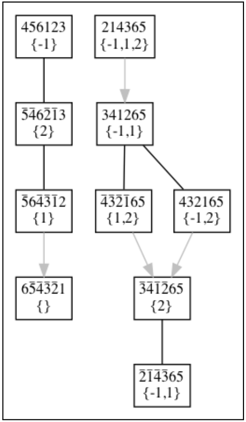


$\Gamma^-(W_3^{BC})$

Gelfand W-graph for W_3^D



$\Gamma^+(W_3^D)$



$\Gamma^-(W_3^D)$

W-graphs duality

In these examples, there is an edge-reversing bijection $\Gamma^+(W) \xrightarrow{\sim} \Gamma^-(W)$. When such a map exists, we say that $\Gamma^+(W)$ and $\Gamma^-(W)$ are *dual*.

Theorem (Marberg–Z. (2020))

The W-graphs $\Gamma^+(W)$ and $\Gamma^-(W)$ are always dual if $W \in \{W_n^{\mathrm{BC}}, W_n^{\mathrm{D}}\}$.

By contrast $\Gamma^+(S_n)$ and $\Gamma^-(S_n)$ do not seem to be dual for any $n > 3$.

The strongly connected components in a W-graph are its *cells*. Each cell inherits its own W-graph structure so defines an $\mathcal{H}(W)$-subrepresentation.

Conjecture. The cells in $\Gamma^+(S_n)$ and $\Gamma^-(S_n)$ are always irreducible.

The cells in $\Gamma^\pm(W_n^{\mathrm{BC}})$ and $\Gamma^\pm(W_n^{\mathrm{D}})$ are not always irreducible. But to understand these, one can just study $\Gamma^+(W_n^{\mathrm{BC}})$ and $\Gamma^+(W_n^{\mathrm{D}})$ via duality.

Type A molecules

- The KL basis $\rightsquigarrow$ a W-graph on W, whose cells are called *left cells*.
- The left cells of S_n are the sets of permutations w with the same recording tableau $Q(w)$ under the RSK correspondence.
- Also for S_n, every left cell is a *molecule*: a set of vertices in a W-graph connected by *bidirected edges* $y \to z$ and $z \to y$.

Theorem (Marberg–Z. (2022+))

The molecules in $\Gamma^+(S_n)$ and $\Gamma^-(S_n)$ are the subsets of $\mathcal{K}(S_n)$ with the same shape under two certain RSK-type insertion algorithms $z \mapsto Q^\pm(z)$.

Conjecture. Every molecule in $\Gamma^+(S_n)$ and $\Gamma^-(S_n)$ is actually a cell.

The relevant algorithms are similar but distinct. There does not appear to be any simple relationship between the cells in $\Gamma^+(S_n)$ and $\Gamma^-(S_n)$.