Approximation and decomposition of clutters

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Based on joint work with Jaume Martí-Farré 2015, 2017, 2022?

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Notation

- Ω is a finite set
- a clutter on Ω is an antichain of subsets of Ω:

$$\Lambda = \{A_1, \ldots, A_n\} \text{ with } A_i \nsubseteq A_j \text{ for all } i \neq j, n \geqslant 0$$

- $\mathrm{Clutt}(\Omega)$ denotes the set of clutters on Ω
- A matroid M consists of a finite set Ω and a clutter $\mathcal{C}(M) \neq \{\varnothing\}$, called the clutter of circuits, s.t.

$$C_1, C_2 \in \mathcal{C}(M), e \in C_1 \cap C_2 \Rightarrow \exists\, C_3 \in \mathcal{C}(M) \text{ s.t. } C_3 \subseteq (C_1 \cup C_2) \backslash e$$

  The subsets of Ω that contain some circuit are called dependent
- A matroid M consists of a finite set Ω and a clutter $\mathcal{B}(M) \neq \{\}$, called the clutter of bases, s.t.

$$B_1, B_2 \in \mathcal{B}(M), x \in B_1 \backslash B_2 \Rightarrow \exists\, y \in B_2 \backslash B_1 \text{ s.t } B_2 \backslash y \cup x \in \mathcal{B}(M)$$

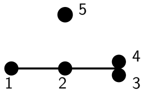
  The subsets of Ω that are contained in some basis are called independent

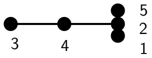
The starting question

Given an arbitrary clutter Λ, which are the matroids "closest" to it?

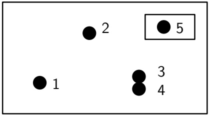
Example: let $\Lambda = \{1\,2\,3, 1\,2\,4, 3\,4\,5\}$

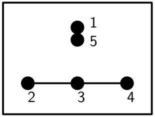
- Both $\{1\,2\,3, 1\,2\,4, 3\,4\}$ and $\{1\,3\,4, 2\,3\,4, 3\,4\,5, 1\,2, 1\,5, 2\,5\}$ are clutters of circuits of a matroid:





- Both $\{1\,2\,3, 1\,2\,4\}$ and $\{1\,2\,3, 1\,2\,4, 3\,4\,5, 1\,3\,4, 2\,3\,5, 2\,4\,5\}$ are clutters of bases of a matroid:





Linking clutters and matroids

- Vaderlind, 1986: *Clutters and semimatroids*
- Dress and Wenzel, 1990: *Matroidizing set systems: a new approach to matroid theory*
- Cordovil, Fukuda, and Moreira, 1991: *Clutters and matroids*
- Traldi, 1997-2003:*Clutters and circuits I, II, III*
- Blasiak, Rowe, Traldi, and Yacobi, 2005: *Several definitions of matroids*
- Martini and Wenzel, 2005: *Symmetrization of closure operators and visibility*
- Martí-Farré, 2014: *From clutters to matroids*

Definitions: clutters

For $\Lambda \in \text{Clutt}(\Omega)$, let

$$\Lambda^+ = \{B \subseteq \Omega : B \supseteq A \text{ for some } A \in \Lambda\}$$
$$\Lambda^- = \{B \subseteq \Omega : B \subseteq A \text{ for some } A \in \Lambda\}$$

Hence

$$\Lambda = \text{minimal}(\Lambda^+) = \text{maximal}(\Lambda^-)$$

Ex: $\mathcal{C}(M)^+ \rightarrow$ dependent sets
$\mathcal{B}(M)^- \rightarrow$ independent sets

Leads to two partial orders on $\text{Clutt}(\Omega)$

$$\Lambda_1 \leqslant^+ \Lambda_2 \iff \Lambda_1^+ \subseteq \Lambda_2^+$$
$$\iff \forall\ A \in \Lambda_1\ \exists\ A' \in \Lambda_2\ s.t.\ A \supseteq A'$$
$$\Lambda_1 \leqslant^- \Lambda_2 \iff \Lambda_1^- \subseteq \Lambda_2^-$$
$$\iff \forall\ A \in \Lambda_1\ \exists\ A' \in \Lambda_2\ s.t.\ A \subseteq A'$$

Definitions: clutters

The clutter Λ^c is $\{\Omega \backslash A : A \in \Lambda\}$

The blocker of a clutter is

$$b(\Lambda) = \text{minimal}\{B : B \cap A \neq \emptyset \text{ for all } A \in \Lambda\}$$

It is well known that $b(b(\Lambda)) = \Lambda$ (Edmonds, Fulkerson 70)

Lem $\Lambda_1 \leqslant^+ \Lambda_2 \Leftrightarrow \Lambda_1^c \leqslant^- \Lambda_2^c$
$\Lambda_1 \leqslant^+ \Lambda_2 \Leftrightarrow b(\Lambda_2) \leqslant^+ b(\Lambda_1)$
$\Lambda_1 \leqslant^- \Lambda_2 \Leftrightarrow b(\Lambda_2^c)^c \leqslant^- b(\Lambda_1^c)^c$

We call $\{\}, \{\emptyset\}$ trivial, and $\{\}, \{\Omega\}$ cotrivial

Definitions: interpretations

For matroids:

Λ	Λ^c	$b(\Lambda)$
$\mathcal{B}(M)$	$\mathcal{B}(M^*)$	$\mathcal{C}(M^*)$
$\mathcal{C}(M)$	$\mathcal{H}(M^*)$	$\mathcal{B}(M^*)$
$\mathcal{H}(M)$	$\mathcal{C}(M^*)$	–

$\mathcal{C}(M_1) \leqslant^+ \mathcal{C}(M_2) \Leftrightarrow M_1$ is above M_2 in the *weak* order

$\mathcal{B}(M_1) \leqslant^- \mathcal{B}(M_2) \Leftrightarrow M_1$ is below M_2 in the *weak* order

$\mathcal{B}(M_1) \leqslant^+ \mathcal{B}(M_2) \Leftrightarrow M_1^*$ is below M_2^* in the *weak* order

Formalizing the initial question

Fix a family $\Sigma \subset \mathrm{Clutt}(\Omega)$ and let $\Lambda \in \mathrm{Clutt}(\Omega)$. Define

$$\begin{aligned}
\Sigma_u^+(\Lambda) &= \{\Lambda' \in \Sigma \,:\, \Lambda \leqslant^+ \Lambda'\} \\
\Sigma_\ell^+(\Lambda) &= \{\Lambda' \in \Sigma \,:\, \Lambda' \leqslant^+ \Lambda\} \\
\Sigma_u^-(\Lambda) &= \{\Lambda' \in \Sigma \,:\, \Lambda \leqslant^- \Lambda'\} \\
\Sigma_\ell^-(\Lambda) &= \{\Lambda' \in \Sigma \,:\, \Lambda' \leqslant^- \Lambda\}
\end{aligned}$$

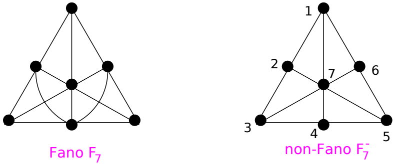
For any choice of $s \in \{u, \ell\}$ and $\square \in \{+, -\}$, is $\Sigma_s^\square(\Lambda) \neq \emptyset$?

Do the minimal clutters of $\Sigma_u^\square(\Lambda)$ (if any) determine Λ?

Do the maximal clutters of $\Sigma_\ell^\square(\Lambda)$ (if any) determine Λ?

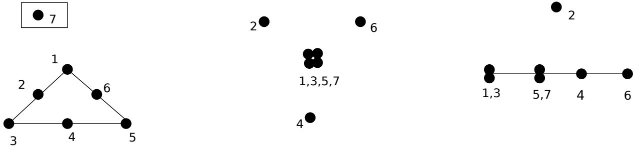
Example

The non-Fano matroid F_7^- is non-binary



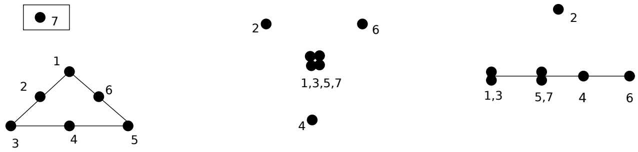
Let $\Sigma = \{\mathcal{C}(M) : M \text{ is binary }\}$

$\Sigma_u^+(\mathcal{C}(F_7^-))$ has 9 minimal elements, consisting of F_7 and 8 matroids, up to isomorphism:



Example

$\Sigma_u^+(\mathcal{C}(F_7^-))$ has 9 minimal elements, consisting of F_7 and 8 matroids, up to isomorphism:



If N is any of these 8 matroids,

$$\mathcal{C}(F_7^-) = \text{minimal}\{C_1 \cup C_2 : C_1 \in \mathcal{C}(N), C_2 \in \mathcal{C}(F_7)\}$$

Informal statement of result

For any choice of $s \in \{u, \ell\}$ and $\square \in \{+, -\}$, there is a family of clutters $\mathcal{F}_s^\square \subset \mathrm{Clutt}(\Omega)$ such that if $\mathcal{F}_s^\square \subseteq \Sigma$, then $\Sigma_s^\square(\Lambda) \neq \varnothing$

Let $\Lambda_1, \ldots, \Lambda_r$ be the elements of Σ closest to Λ with respect to $\square$ and s (the optimal completions)

There is an operation $\star$ such that $\Lambda = \Lambda_1 \star \cdots \star \Lambda_r$ (the decomposition)

Lattice structure

$\mathrm{Clutt}(\Omega)^+ = (\mathrm{Clutt}(\Omega), \leqslant^+, \sqcap^+, \sqcup^+)$ is a distributive lattice:

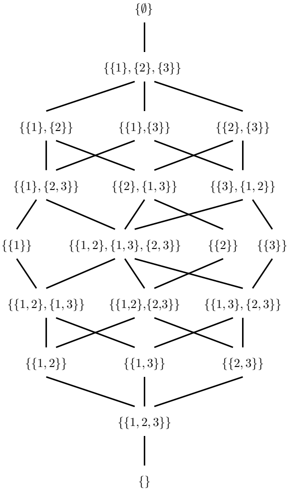
$$\Lambda_1 \sqcap^+ \Lambda_2 = \min(\Lambda_1^+ \cap \Lambda_2^+) = \min\{A_1 \cup A_2 : A_i \in \Lambda_i\}$$
$$\Lambda_1 \sqcup^+ \Lambda_2 = \min(\Lambda_1^+ \cup \Lambda_2^+) = \min(\Lambda_1 \cup \Lambda_2)$$

$\mathrm{Clutt}(\Omega)^- = (\mathrm{Clutt}(\Omega), \leqslant^-, \sqcap^-, \sqcup^-)$ is a distributive lattice:

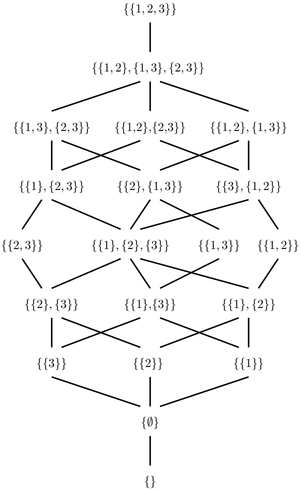
$$\Lambda_1 \sqcap^- \Lambda_2 = \max(\Lambda_1^- \cap \Lambda_2^-) = \max\{A_1 \cap A_2 : A_i \in \Lambda_i\}$$
$$\Lambda_1 \sqcup^- \Lambda_2 = \max(\Lambda_1^- \cup \Lambda_2^-) = \max(\Lambda_1 \cap \Lambda_2)$$

- $(\Lambda_1 \sqcap^+ \cdots \sqcap^+ \Lambda_r)^c = \Lambda_1^c \sqcap^- \cdots \sqcap^- \Lambda_r^c$
- $(\Lambda_1 \sqcup^+ \cdots \sqcup^+ \Lambda_r)^c = \Lambda_1^c \sqcup^- \cdots \sqcup^- \Lambda_r^c$
- $b(\Lambda_1 \sqcap^+ \cdots \sqcap^+ \Lambda_r) = b(\Lambda_1) \sqcup^+ \cdots \sqcup^+ b(\Lambda_r)$
- $b(\Lambda_1 \sqcup^+ \cdots \sqcup^+ \Lambda_r) = b(\Lambda_1) \sqcap^+ \cdots \sqcap^+ b(\Lambda_r)$

The free distributive lattices on clutters



$(\mathrm{Clutt}(\{1,2,3\}), \leq^+)$



$(\mathrm{Clutt}(\{1,2,3\}), \leq^-)$

Σ-meet decomposition of Λ:
$\Lambda = \Lambda_1 \wedge \cdots \wedge \Lambda_k$ for $\Lambda_i \in \Sigma$, $\wedge \in \{\sqcap^+, \sqcap^-\}$

Σ-join decomposition of Λ:
$\Lambda = \Lambda_1 \vee \cdots \vee \Lambda_k$ for $\Lambda_i \in \Sigma$, $\vee \in \{\sqcup^+, \sqcup^-\}$

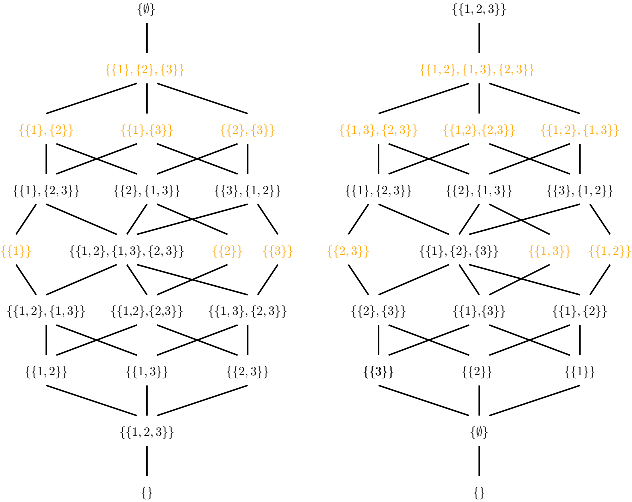
There is a Σ-meet decomposition in $\mathrm{Clutt}(\Omega)^+$ for all non-trivial Λ if and only if $\mathcal{M}(\mathrm{Clutt}(\Omega)^+) \subseteq \Sigma$
(where $\mathcal{M}(\mathrm{Clutt}(\Omega)^+)$ are the meet-irreducible elements)

$$\mathcal{M}(\mathrm{Clutt}(\Omega)^+) = \{\{\{a_1\}, \{a_2\}, \ldots, \{a_i\}\} : \{a_1, \ldots, a_i\} \subseteq \Omega, i \geqslant 1\}$$

Also,

$$\mathcal{J}(\mathrm{Clutt}(\Omega)^+) = \{\{a_1, a_2, \ldots, a_i\}\} : \{a_1, \ldots, a_i\} \subseteq \Omega, i \geqslant 1\}$$
$$\mathcal{M}(\mathrm{Clutt}(\Omega)^-) = \{\{\Omega \backslash a_1, \Omega \backslash a_2, \ldots, \Omega \backslash a_i\} : \{a_1, \ldots, a_i\} \subseteq \Omega, i \geqslant 1\}$$
$$\mathcal{J}(\mathrm{Clutt}(\Omega)^-) = \{\{a_1, a_2, \ldots, a_i\}\} : \{a_1, \ldots, a_i\} \subseteq \Omega, 0 \leqslant i < |\Omega|\}$$

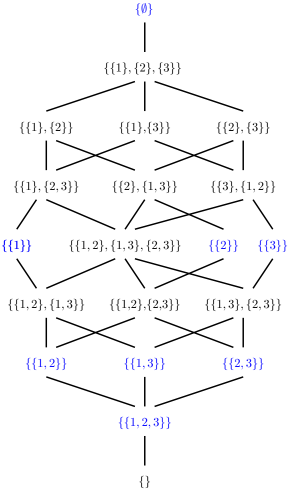
Meet-irreducible elements



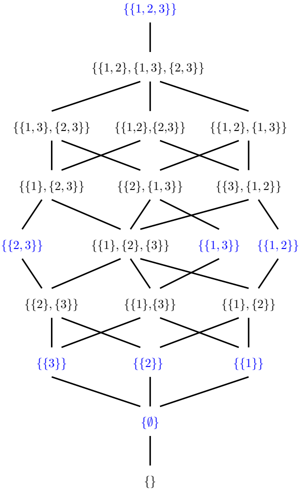
(Clutt({1, 2, 3}), $\leq^+$)

(Clutt({1, 2, 3}), $\leq^-$)

Join-irreducible elements



$(\mathrm{Clutt}(\{1,2,3\}), \leq^{+})$



$(\mathrm{Clutt}(\{1,2,3\}), \leq^{-})$

Summary

$$\mathcal{M}(\mathrm{Clutt}(\Omega)^+) = \{\{\{a_1\},\{a_2\},\ldots,\{a_i\}\} : \{a_1,\ldots,a_i\} \subseteq \Omega, i \geqslant 1\}$$
$$\mathcal{J}(\mathrm{Clutt}(\Omega)^+) = \{\{a_1,a_2,\ldots,a_i\}\} : \{a_1,\ldots,a_i\} \subseteq \Omega, i \geqslant 1\}$$
$$\mathcal{M}(\mathrm{Clutt}(\Omega)^-) = \{\{\Omega\backslash a_1,\Omega\backslash a_2,\ldots,\Omega\backslash a_i\} : \{a_1,\ldots,a_i\} \subseteq \Omega, i \geqslant 1\}$$
$$\mathcal{J}(\mathrm{Clutt}(\Omega)^-) = \{\{a_1,a_2,\ldots,a_i\}\} : \{a_1,\ldots,a_i\} \subseteq \Omega, 0 \leqslant i < |\Omega|\}$$

Hence, for a non-trivial, non-cotrivial clutter Λ

- $\Sigma_u^\square(\Lambda) \neq \varnothing$ and Λ can be recovered from minimal$(\Sigma_u^\square(\Lambda))$ if and only if $\mathcal{M}(\mathrm{Clutt}(\Omega)^\square \subseteq \Sigma$
- $\Sigma_\ell^\square(\Lambda) \neq \varnothing$ and Λ can be recovered from maximal$(\Sigma_\ell^\square(\Lambda))$ if and only if $\mathcal{J}(\mathrm{Clutt}(\Omega)^\square \subseteq \Sigma$

The matroid case

- $\{\{\{a_1\},\{a_2\},\ldots,\{a_i\}\} : \{a_1,\ldots,a_i\} \subseteq \Omega, i \geqslant 1\}$
  they are clutters of circuits (i loops, the others coloops) and of bases (i elements in parallel, the others loops)
- $\{\{a_1,a_2,\ldots,a_i\}\} : \{a_1,\ldots,a_i\} \subseteq \Omega, i \geqslant 1\}$
  they are clutters of circuits (i-circuit, the others coloops) and of bases (i coloops, the others loops)
- $\{\{\Omega \backslash a_1, \Omega \backslash a_2, \ldots, \Omega \backslash a_i\} : \{a_1,\ldots,a_i\} \subseteq \Omega, i \geqslant 1\}$
  they are clutters of bases (i-circuit, the others coloops), but not of circuits in general

Finding matroidal completions: reductions

Let $\mathfrak{C} = \{\mathcal{C}(M) : M \text{ is a matroid on } \Omega\}$
$\mathfrak{B} = \{\mathcal{B}(M) : M \text{ is a matroid on } \Omega\}$

The only case in which a decomposition is not ensured is $\mathfrak{C}_u^-$

By combining blockers and complements, it is enough to solve one case in each row (theoretically)

- $\mathfrak{C}_u^+, \mathcal{B}_\ell^+, \mathcal{B}_\ell^-$
- $\mathfrak{C}_\ell^+, \mathcal{B}_u^+, \mathcal{B}_u^-$
- $\mathfrak{C}_\ell^-$

The case $\mathfrak{C}_u^+$ is from Martí-Farré 2014:
Given Λ, an algorithm produces the minimal $\Lambda_1, \ldots, \Lambda_r$ such that

$$\Lambda \leqslant^+ \Lambda_i \text{ and } \Lambda_i \in \mathfrak{C}$$

(Dress, Wenzel 90 constructed M s.t. $\mathcal{B}(M) \leqslant^+ \Lambda$)

Finding the completions: algorithms

Similarly, $\mathfrak{C}_\ell^+$ and $\mathfrak{C}_\ell^-$ are in Martí-Farré, dM 17

All three algorithms for circuits completions use the following:

For distinct $A_1, A_2 \in \Lambda$, define

$$I_\Lambda(A_1 \cup A_2) = \bigcap_{X \in \Lambda, X \subseteq A_1 \cup A_2} X$$

Lem $\Lambda \in \mathfrak{C} \Leftrightarrow I_\Lambda(A_1 \cup A_2) = \emptyset$ for all $A_1 \neq A_2 \in \Lambda$

Finding the completions: algorithms

Let's look at $\mathfrak{C}_\ell^+$:

For a clutter Λ and a set $A \in \Lambda$,

- If there is $A' \in \Lambda$ with $A \neq A'$ and $I_\Lambda(A \cup A') \neq \emptyset$, then

$$\beta(\Lambda; A) = \text{minimal}\left(\Lambda \backslash \{A\} \cup \{A \cup x : x \in \Omega \backslash A\}\right)$$

- Otherwise, set $\beta(\Lambda; A) = \Lambda$.

Lem

$\beta(\Lambda; A) \leqslant^+ \Lambda$, with strict inequality if the first case of the definition applies

If $\beta(\Lambda; A) = \Lambda$ for all A, then Λ is a clutter of circuits

Finding the completions: algorithms

Thm Let Λ' be an optimal completion of Λ for $\mathfrak{C}_\ell^+$. Then there exist $\Lambda_1, \ldots, \Lambda_k$ s.t.

$$\Lambda' = \Lambda_k \leqslant^+ \Lambda_{k-1} \leqslant^+ \cdots \leqslant^+ \Lambda_1 \leqslant^+ \Lambda_0 = \Lambda$$

with $\Lambda_i = \beta(\Lambda_{i-1}; A^{(i-1)})$ for $1 \leqslant i \leqslant k$

So to obtain all optimal completions, we apply the β transformations in all possible orders until no more applications are possible, and take the maximal clutters thus generated

Finding the completions: example

Let $\Lambda = \{123, 124\}$; $I_\Lambda(123 \cup 124) = \{1\} \neq \varnothing$
Then:

- $\Lambda_1 = \beta(\Lambda, 123) = \text{minimal}\{124, 1234, 1235\} = \{1235, 124\}$
- $\Lambda_2 = \beta(\Lambda, 124) = \text{minimal}\{123, 1234, 1245\} = \{1234, 123\}$
- $\Lambda_3 = \beta(\Lambda_1; 1235) = \{124\}$; $\Lambda_4 = \beta(\Lambda_2; 1234) = \{123\}$
- $\Lambda_5 = \beta(\Lambda_1; 124) = \{1235, 1234, 1245\}$
- As long as the clutter has more than one set, it won't be a circuit clutter (because of the 1). So the β transformation will remove sets until one is left, and the result will be always $\leqslant^- \Lambda_3$ or $\leqslant^- \Lambda_4$

Clearly $\{123, 124\} = \{123\} \sqcup^+ \{124\}$

Finding the completions: example

For the other two types of circuit completions we have similar looking algorithms

For instance, the case $\mathfrak{C}_\ell^-$ gives as optimal completions of $\{123, 124\}$

$$\{123, 4\}, \{124, 3\}, \{13, 24\}, \{14, 23\}$$

although

$$\{123, 124\} = \{123, 4\} \sqcup^- \{124, 3\}$$

Fixing the ground set

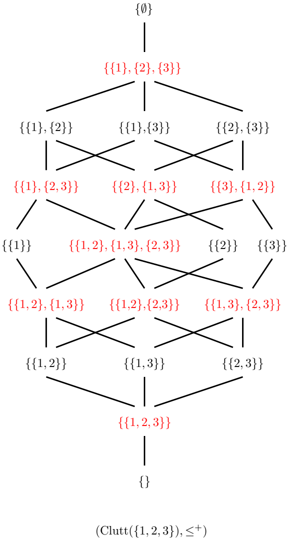
Let $\mathrm{Clutt}(\Omega)_0 = \{\Lambda \in \mathrm{Clutt}(\Omega) : \bigcup_{A \in \Lambda} A = \Omega\}$

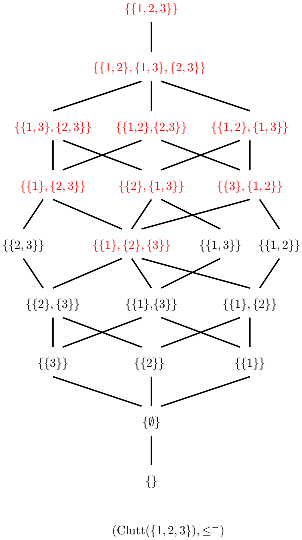
- $\mathfrak{C} \nsubseteq \mathrm{Clutt}(\Omega)_0$, $\mathfrak{B} \nsubseteq \mathrm{Clutt}(\Omega)_0$
- But one may want to look at loop and coloop free matroids
- Several clutters associated to graphs belong to $\mathrm{Clutt}(\Omega)_0$

(Martí-Farré, Mora, Ruiz 2018)

If $\Lambda \in \mathrm{Clutt}(\Omega)_0, \Sigma \subseteq \mathrm{Clutt}(\Omega)_0$, can we achieve analogous completion/decomposition results?

The posets $\text{Clutt}(\Omega)_0^+, \text{Clutt}(\Omega)_0^-$





The posets $\mathrm{Clutt}(\Omega)_0^+, \mathrm{Clutt}(\Omega)_0^-$

If $\Omega = \{a_1, \dots, a_n\}$

- $\mathrm{Clutt}(\Omega)_0^-$ is a sublattice of $\mathrm{Clutt}(\Omega)^-$

$$\mathrm{Clutt}(\Omega)_0 = \{\Lambda \in \mathrm{Clutt}(\Omega) : \{\{a_1\}, \dots, \{a_n\}\} \leqslant^- \Lambda\}$$

  and

$$\begin{aligned} \mathcal{M}(\mathrm{Clutt}(\Omega)_0^-) &= \mathcal{M}(\mathrm{Clutt}(\Omega)^-) \cap \mathrm{Clutt}(\Omega)_0 \\ &= \{\{\Omega \backslash a : a \in A\} : A \subseteq \Omega, |A| \geqslant 2\} \\ \mathcal{J}(\mathrm{Clutt}(\Omega)_0^-) &= \{\{A_1, \dots, A_k\} : A_1, \dots, A_k \text{ is a partition of } \Omega\} \end{aligned}$$

- $\mathrm{Clutt}(\Omega)_0^+$ does not have a lattice structure in general ($\{12, 13, 14\}$ and $\{13, 34, 14\}$ have two maximal upper bounds, $\{123, 14\}$ and $\{123, 15\}$)
- $\Lambda \in \mathrm{Clutt}(\Omega)_0 \Rightarrow b(\Lambda) \in \mathrm{Clutt}(\Omega)_0$

The case $\mathrm{Clutt}(\Omega)_0{}^+$

Let $\mathcal{M}_0^+$ be the collection of clutters

- $\{\{a_1\}, \ldots, \{a_n\}\}$
- $\{\{a_{\sigma(1)}, a_{\sigma(2)}\}, \ldots, \{a_{\sigma(1)}, a_{\sigma(n)}\}\}$ for each $\sigma \in \mathfrak{S}_n$
- $\{\{a_{\sigma(1)}\}, \ldots, \{a_{\sigma(r)}\}, \{a_{\sigma(r+1)}, a_{\sigma(r+2)}\} \ldots, \{a_{\sigma(r+1)}, a_{\sigma(n)}\}\}$ for each $\sigma \in \mathfrak{S}_n$, $1 \leqslant r \leqslant n-2$

Thm For $\Sigma \subseteq \mathrm{Clutt}(\Omega)_0$, every clutter $\Lambda \in \mathrm{Clutt}(\Omega)_0$ admits a Σ-meet decomposition in $\mathrm{Clutt}(\Omega)^+$ if and only if $\mathcal{M}_0^+ \subseteq \Sigma$

Thm For $\Sigma \subseteq \mathrm{Clutt}(\Omega)_0$, every clutter $\Lambda \in \mathrm{Clutt}(\Omega)_0$ admits a Σ-join decomposition in $\mathrm{Clutt}(\Omega)^+$ if and only if $b(\mathcal{M}_0^+) \subseteq \Sigma$

The case $\mathrm{Clutt}(\Omega)_0{}^+$, proof

Thm For $\Sigma \subseteq \mathrm{Clutt}(\Omega)_0$, every clutter $\Lambda \in \mathrm{Clutt}(\Omega)_0$ admits a Σ-meet decomposition in $\mathrm{Clutt}(\Omega)^+$ if and only if $\mathcal{M}_0^+ \subseteq \Sigma$

- Clutters in $\mathcal{M}_0^+$ are $\sqcap^+$ irreducible in $\mathrm{Clutt}(\Omega)_0$
- Given $\Lambda \in \mathrm{Clutt}(\Omega)_0$, let $\{\Lambda_1, \ldots, \Lambda_k\}$ be the minimal elements of Σ above Λ (there is always some)
  Let $\Lambda_0 = \Lambda_1 \sqcap^+ \cdots \sqcap^+ \Lambda_k$
  - clearly $\Lambda \leqslant^+ \Lambda_0$; assume $\Lambda_0 \not\leqslant^+ \Lambda$
  - there is $A = \{a_1, \ldots, a_r\} \in \Lambda_0$ s.t. no subset of A is in Λ
  - since Λ covers Ω, wlog, there is $B \in \Lambda$ s.t. $\{a_1, a_{r+1}\} \subseteq B \in \Lambda$
  - let $\widetilde{\Lambda} = \{\{a_1, a_{r+1}\}, \ldots, \{a_r, a_{r+1}\}, \{a_{r+2}\}, \{a_n\}\} \in \Sigma$
  - $\Lambda \leqslant^+ \widetilde{\Lambda}$, so $\Lambda_0 \leqslant^+ \widetilde{\Lambda}$
  - but $A \in \Lambda_0$ and for all $C \in \widetilde{\Lambda}$, $C \nsubseteq A$, a contradiction

A few questions

- Can one do better in practice finding the matroidal completions?
- Clutters that have exactly two optimal matroidal completions should be the most "matroid like"
- Is there an analogue of "covering the ground set" for an arbitrary distributive lattice?

THANK YOU !