On Framed Triangulations of Flow Polytopes, the ν-Tamari Lattice and Young's Lattice

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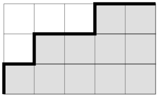
FPSAC'2021 Ramat-Gan, Israel

ν-Dyck paths

Definition

Let ν be a lattice path from $(0,0)$ to (a,b) with steps $N=(0,1)$, $E=(1,0)$.
A ν-**Dyck** path is a lattice path which stays weakly above ν.

A path ν



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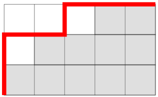

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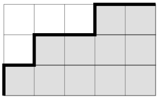

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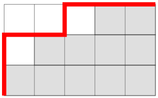

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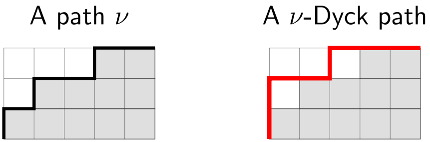

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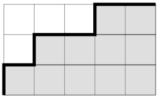

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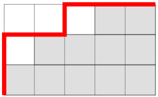

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Choosing $\nu=(NE)^n$ recovers the classical Dyck paths.

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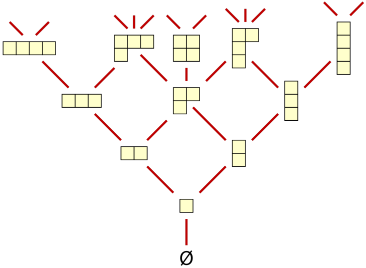

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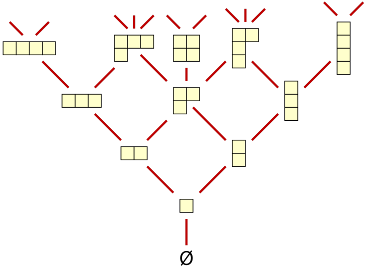

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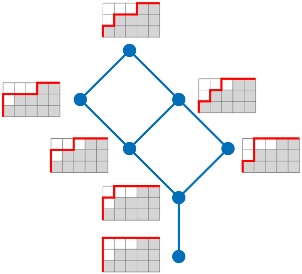

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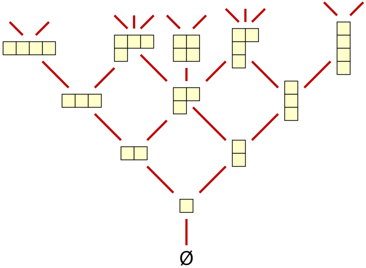

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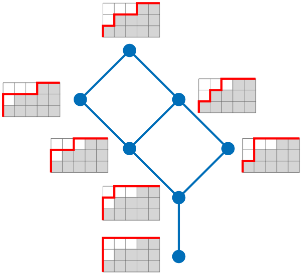

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Cover relation in $I(\nu)$: $\pi_1 \prec \pi_2 \Leftrightarrow \pi_2$ obtained from π_1 by $NE \to EN$

Another lattice structure on ν-Dyck paths

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The **horizontal distance** of a point p on a ν-Dyck path is

$$\mathrm{horiz}_\nu(p) = \# \text{ of } E \text{ steps that can be taken from } p \text{ without crossing } \nu.$$

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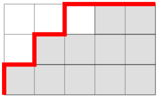

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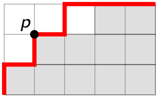

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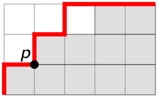

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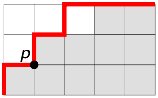

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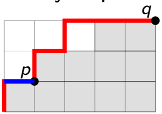

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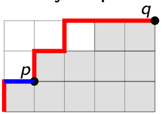

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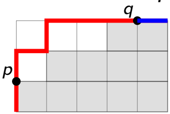

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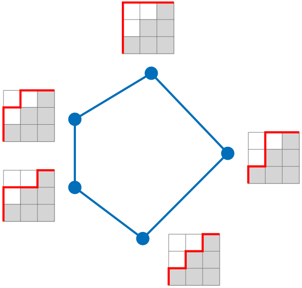
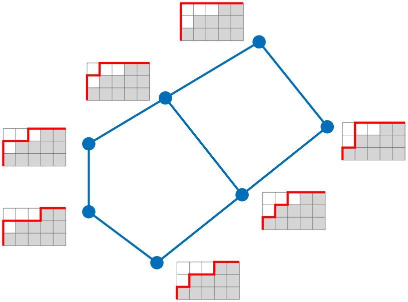
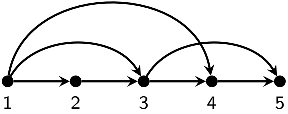




Flow polytopes

A **flow graph** G is a connected digraph on vertex set $V = \{1, 2, ..., n\}$ with edge multiset E, where all edges are directed from smaller to larger vertices.



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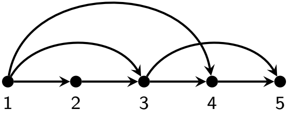

A **flow of size one** on G is a tuple $(x_e)_{e \in E(G)}$ of non-negative real numbers such that flow is preserved at each vertex, i.e.

$$\sum_{e \in \mathrm{Out}(1)} x_e = \sum_{e \in \mathrm{In}(n)} x_e = 1, \text{ and } \sum_{e \in \mathrm{In}(i)} x_e - \sum_{e \in \mathrm{Out}(i)} x_e = 0 \text{ for } i \in [2, n-1].$$

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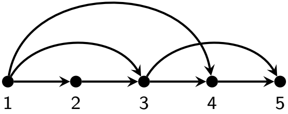

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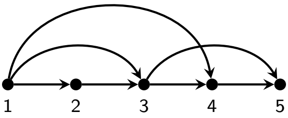

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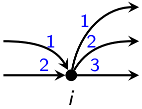

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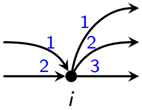

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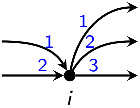

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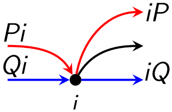

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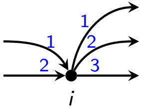

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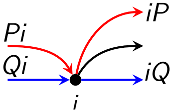

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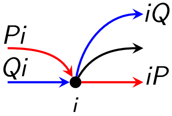

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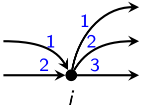

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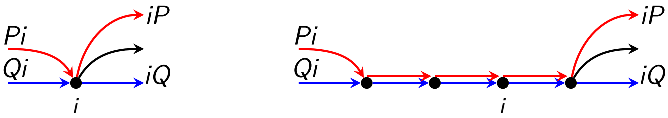

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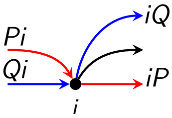

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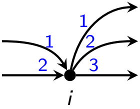

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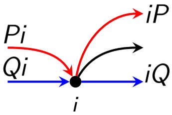

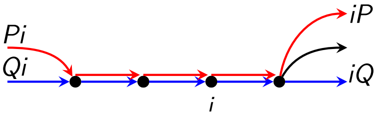

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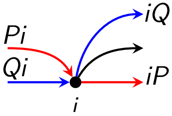

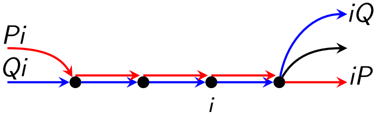

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Theorem (Danilov–Karzanov–Koshevoy, 2012)

The maximal sets of coherent routes in a framing of G determine simplices in a regular unimodular triangulation of $\mathcal{F}_G$.

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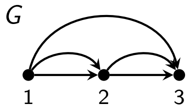

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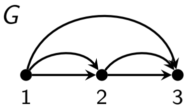

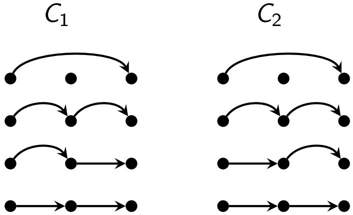

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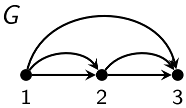
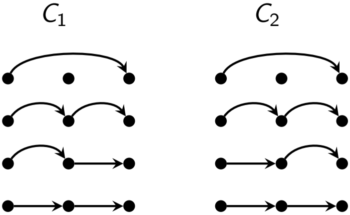


$$\mathcal{F}_G = \Delta_{C_1} \cup \Delta_{C_2}$$



The ν-caracol flow polytope

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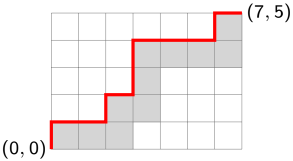

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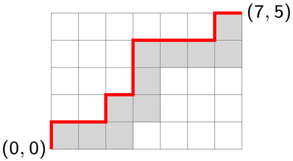

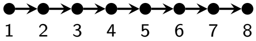

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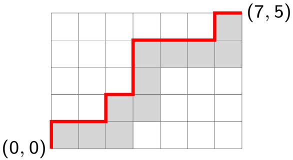

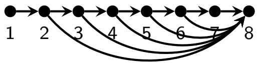

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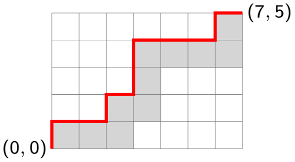

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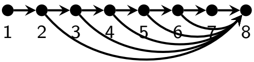

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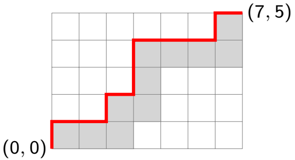

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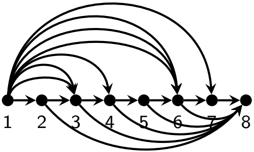

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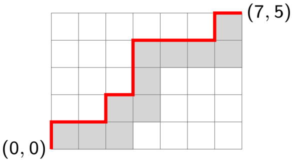

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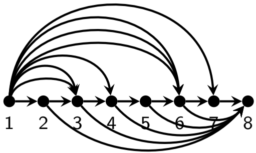

$\mathcal{F}_{\mathrm{car}(\nu)} = \nu$-**caracol flow polytope**

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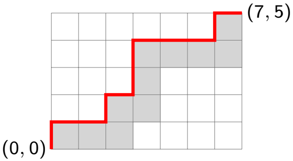

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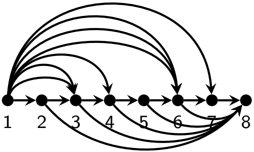

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Mészáros-Morales, 2019:

$$\text{vol}(\mathcal{F}_{\text{car}(\nu)}) = \det\left(\binom{1+\sum_{k=1}^{a-j}\nu_k}{1+j-i}\right)_{1\le i,j\le a-1} = \#\{\nu\text{-Dyck paths}\}$$

The planar-framed triangulation

Theorem (B.–González D'León–Mayorga Cetina–Yip, 2021)

The dual graph of the planar-framed triangulation of $\mathcal{F}_{\text{car}(\nu)}$ is the Hasse diagram of the principal order ideal $I(\nu)$ in Young's lattice.

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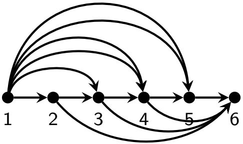

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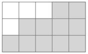

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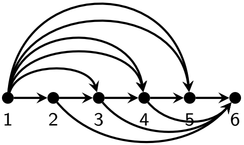

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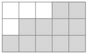

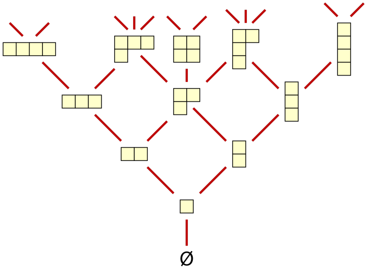

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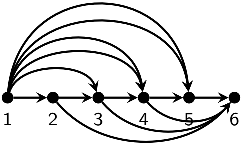

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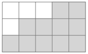
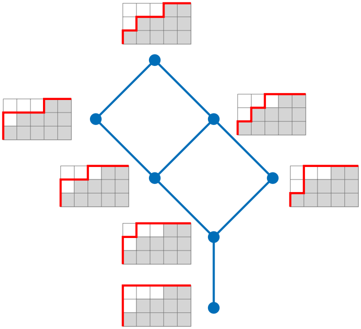


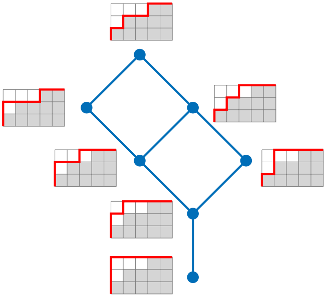


h^*-vector of $\mathcal{F}_{\text{car}(\nu)}$

Theorem (B.–González D'León–Mayorga Cetina–Yip, 2021)

The h^*-polynomial of $\mathcal{F}_{\text{car}(\nu)}$ is the ν-Narayana polynomial

$$N_\nu(x) := \sum_{i \geq 0} \text{Nar}(\nu, i) x^i.$$



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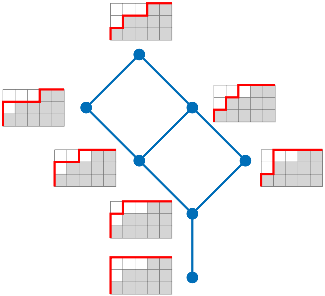

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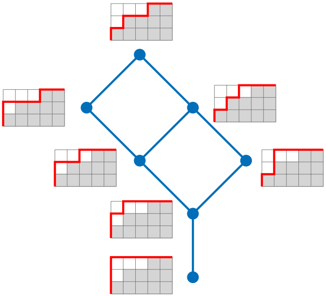

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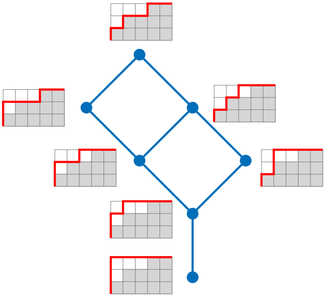

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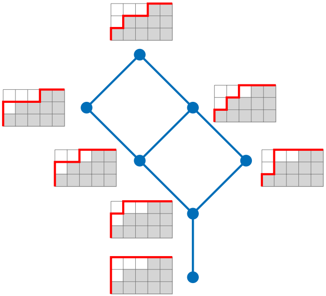

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$$\begin{aligned} h_i &= \#\text{paths in } I(\nu) \text{ covering } i \text{ paths} \\ &= \#\text{paths with } i \text{ valleys} \\ &= \mathrm{Nar}(\nu, i) \end{aligned}$$

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Theorem (B.–González D'León–Mayorga Cetina–Yip, 2021)

The dual graph of the length-framed triangulation of $\mathcal{F}_{\mathsf{car}(\nu)}$ *is the Hasse diagram of the* ν*-Tamari lattice* $\mathrm{Tam}(\nu)$.

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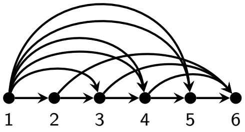

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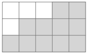

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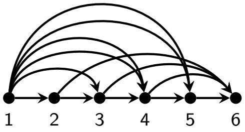

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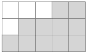

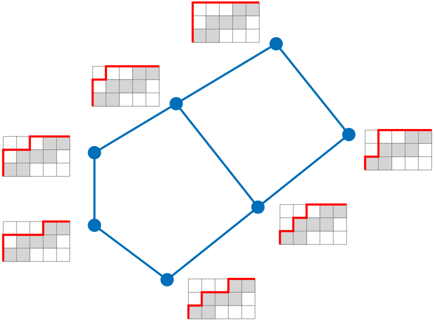

Préville-Ratelle–Viennot, 2017

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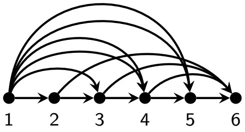

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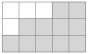

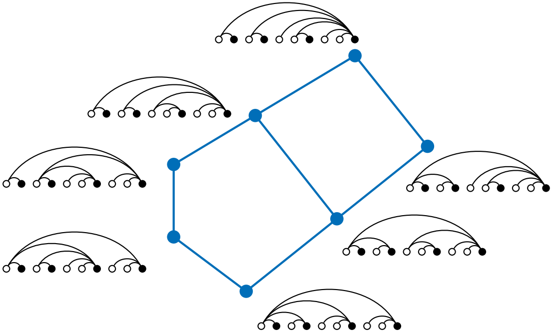

Ceballos-Padrol-Sarmiento, 2019

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- **Question:** Are there any other interesting lattice/graph structures on ν-Catalan objects obtainable using different framings?

Thank you!

References

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